

Classifying uncountable structures up to bi-embeddability

Alessandro Andretta

Luca Motto Ros

Author address:

UNIVERSITÀ DI TORINO, DIPARTIMENTO DI MATEMATICA, VIA CARLO ALBERTO, 10,
10123 TORINO, ITALY

E-mail address: `alessandro.andretta@unito.it`

UNIVERSITÀ DI TORINO, DIPARTIMENTO DI MATEMATICA, VIA CARLO ALBERTO, 10,
10123 TORINO, ITALY

E-mail address: `luca.mottoros@unito.it`

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Abstract

We provide analogues of the results from [FMR11, CMMR13] (which correspond to the case $\kappa = \omega$) for arbitrary κ -Souslin quasi-orders on any Polish space, for κ an infinite cardinal smaller than the cardinality of \mathbb{R} . These generalizations yield to a variety of results concerning the complexity of the embeddability relation between graphs or lattices of size κ , the isometric embeddability relation between complete metric spaces of density character κ , and the linear isometric embeddability relation between (real or complex) Banach spaces of density κ .

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1. Introduction

1.1. What we knew.

1.1.1. *Equivalence relations and classification problems.* The analysis of definable equivalence relations on Polish spaces (or, more generally, on standard Borel spaces) has been one of the most active areas in descriptive set theory for the last two decades — see [FS89, Kec99, Kan08, Gao09] for excellent surveys of the subject. The main goal of this research area is to classify equivalence relations by means of reductions: if E and F are equivalence relations on Polish or standard Borel spaces X and Y , then $f: X \rightarrow Y$ reduces E to F if and only if

$$(1.1) \quad x E x' \Leftrightarrow f(x) F f(x'),$$

for every $x, x' \in X$. In order to obtain nontrivial results one usually imposes definability assumptions on E and F and/or on the reduction f . For example one may assume that

- E and F are Borel or analytic¹ and f is continuous or Borel (as it is customary when studying actions of Polish groups on standard Borel spaces [BK96, Hjo00, KM04]), or that
- E and F are projective and f is Borel [HS79], or that
- E , F , and f belong to some inner model of determinacy, such as $L(\mathbb{R})$ [Hjo95, Hjo00].

When the reducing function f is Borel we say that E is *Borel reducible* to F (in symbols $E \leq_{\mathbf{B}} F$) and that f is a *Borel reduction* of E to F . If $E \leq_{\mathbf{B}} F \leq_{\mathbf{B}} E$, then E and F are said to be *Borel bi-reducible*, in symbols $E \sim_{\mathbf{B}} F$.

Borel reducibility is usually interpreted both as a topological version of the ubiquitous notion of classification of mathematical objects, and as a tool for computing cardinalities of quotient spaces in an effective way. If E and F are equivalence relations, then it can be argued that the statement $E \leq_{\mathbf{B}} F$ is a precise mathematical formulation of the following informal assertions:

- the problem of classifying the elements of X up to E -equivalence is no more complex than the problem of classifying the elements of Y up to F -equivalence;
- the elements of X can be classified (in a definable way) up to E -equivalence using the F -equivalence classes as invariants;
- the quotient space X/E has cardinality less or equal than the cardinality of X/F , and this inequality can be witnessed in a concrete and definable way.

The theory of Borel reducibility has been used to gauge the complexity of many natural problems. One striking example is given by the classification of countable structures up to isomorphism briefly described below, whose systematic study was initiated by H. Friedman and Stanley in [FS89]. In a pioneering work in the mid 70s [Vau75] Vaught observed that by identifying the universe of a countable structure with ω , the collection $\text{Mod}_{\mathcal{L}}^{\omega}$ of countable \mathcal{L} -structures can be construed as a Polish space, and the isomorphism relation \cong on this space is an analytic equivalence relation. In order to exploit this identification most effectively, first-order logic must be replaced by its infinitary version $\mathcal{L}_{\omega_1\omega}$, where countable conjunctions and disjunctions are allowed [Kei71]: by a theorem of Lopez-Escobar the $(\mathcal{L}_{\omega_1\omega}\text{-})$ elementary classes, that is the sets $\text{Mod}_{\sigma}^{\omega}$ of all countable models of an $\mathcal{L}_{\omega_1\omega}$ -sentence σ , are exactly the Borel subsets of $\text{Mod}_{\mathcal{L}}^{\omega}$ which are invariant under isomorphism. It follows that each elementary class is a standard Borel subspace of $\text{Mod}_{\mathcal{L}}^{\omega}$, and that the restriction of the isomorphism relation to $\text{Mod}_{\sigma}^{\omega}$, denoted in this paper either by $\cong \upharpoonright \text{Mod}_{\sigma}^{\omega}$ or by \cong_{σ}^{ω} , is an analytic equivalence relation, whose complexity can then be analyzed in terms of Borel reducibility.

Other classification problems whose complexity with respect to $\leq_{\mathbf{B}}$ has been widely studied in the literature include e.g. the classification of Polish metric spaces up to isometry [CGK01, GK03, Cle12, CMMR] and the classification of separable Banach spaces up to linear isometry [Mel07] or up to isomorphism [FLR09].

¹As customary in descriptive set theory, analytic sets are also called Σ_1^1 .

1.1.2. *Quasi-orders and embeddability.* The concept of reduction from (1.1) can be applied to arbitrary binary relations, such as partial orders and quasi-orders (also known as preorders) [HMS88]. Quasi-orders are reflexive and transitive relations, and their symmetrization gives rise to an equivalence relation. For example, the embeddability relation \sqsubseteq between structures (graphs, combinatorial trees, lattices, quasi-orders, partial orders, ...) is a quasi-order, and its symmetrization is the relation \approx of bi-embeddability. The restriction of \sqsubseteq to the Borel set Mod_σ^ω , denoted by $\sqsubseteq \upharpoonright \text{Mod}_\sigma^\omega$ or by $\sqsubseteq_\sigma^\omega$, is an analytic quasi-order. Similarly, the relation of bi-embeddability on Mod_σ^ω , denoted by $\approx \upharpoonright \text{Mod}_\sigma^\omega$ or \approx_σ^ω , is an analytic equivalence relation.

In [LR05, Theorem 3.1] Louveau and Rosendal proved that it is not possible to classify (in a reasonable way) all countable structures up to bi-embeddability, as the embeddability relation \sqsubseteq is as complex as possible with respect to $\leq_{\mathbf{B}}$ (whence also \approx is as complex as possible).

THEOREM 1.1 (Louveau-Rosendal). *The embeddability relation $\sqsubseteq_{\text{CT}}^\omega$ on countable combinatorial trees (i.e. connected acyclic graphs) is a $\leq_{\mathbf{B}}$ -complete analytic quasi-order, that is:*

- (a) $\sqsubseteq_{\text{CT}}^\omega$ is an analytic quasi-order on CT_ω , the Polish space of countable combinatorial trees, and
- (b) every analytic quasi-order is Borel reducible to $\sqsubseteq_{\text{CT}}^\omega$.

Thus also the bi-embeddability relation $\approx_{\text{CT}}^\omega$ on CT_ω is a $\leq_{\mathbf{B}}$ -complete analytic equivalence relation.

REMARK 1.2. In an abstract setting, it is not hard to find $\leq_{\mathbf{B}}$ -complete quasi-orders and equivalence relations. In fact it is easy to show that for every pointclass Γ closed under Borel preimages, countable unions, and projections, the collection of quasi-orders (respectively: equivalence relations) in Γ admits a $\leq_{\mathbf{B}}$ -complete element, i.e. there is a quasi-order (respectively, an equivalence relation) $\mathcal{U}_\Gamma \in \Gamma$ such that $R \leq_{\mathbf{B}} \mathcal{U}_\Gamma$, for any quasi-order (respectively: equivalence relation) $R \in \Gamma$ — see [LR05, Proposition 1.3] and the ensuing remark. Examples of Γ as above are the projective classes Σ_n^1 's and $\mathcal{S}(\kappa)$, the collection of all κ -Souslin sets — see Definition 9.1. However such $\leq_{\mathbf{B}}$ -complete \mathcal{U}_Γ 's are usually obtained by *ad hoc* constructions. In contrast, Theorem 1.1 provided the first concrete, natural example of a $\leq_{\mathbf{B}}$ -complete analytic equivalence relation: the relation $\approx_{\text{CT}}^\omega$ of bi-embeddability on combinatorial trees.²

A $\leq_{\mathbf{B}}$ -complete analytic equivalence relation must contain a non-Borel equivalence class, since it reduces the equivalence relation

$$E_A := \{(x, y) \in {}^\omega 2 \times {}^\omega 2 \mid x = y \vee x, y \in A\},$$

where $A \subseteq {}^\omega 2$ is a proper analytic set. On the other hand, the equivalence classes of an equivalence relation E_G induced by a continuous (or Borel) action of a Polish group G are Borel [Kec95, Theorem 15.14], even when E_G is Σ_1^1 and not Borel. Therefore there is a striking difference between the isomorphism relation \cong (which is induced by a continuous action of the group $\text{Sym}(\omega)$ of all permutations on the natural numbers) and the embeddability relation \sqsubseteq : even a very simple relation like the E_A above is not Borel reducible to \cong , while any Σ_1^1 equivalence relation (in fact: any Σ_1^1 quasi-order) is Borel reducible to \sqsubseteq .

Hjorth isolated a topological property, called *turbulence*, that characterizes when an equivalence relation induced by a continuous Polish group action is Borel reducible to \cong [Hjo00].

The next example shows that $\sqsubseteq_{\text{CT}}^\omega$ is also complete for partial orders of size \aleph_1 , in the sense that each such partial order embeds into the quotient order of $\sqsubseteq_{\text{CT}}^\omega$.

EXAMPLE 1.3. Let \preceq be the Σ_2^0 quasi-order on ${}^\omega 2$ induced by inclusion on $\mathcal{P}(\omega)/\text{Fin}$, i.e. $x \preceq y \Leftrightarrow \exists n \forall m \geq n (x(m) \leq y(m))$. Then \preceq can be embedded into $\sqsubseteq_{\text{CT}}^\omega$ by Theorem 1.1. By

²In [FLR09] it is shown that the isomorphism relations between separable Banach spaces, and between Polish groups are other natural examples of $\leq_{\mathbf{B}}$ -complete analytic equivalence relations.

Parovičenko's theorem [Par63], under the Axiom of Choice AC any partial order P of size \aleph_1 embeds into $\mathcal{P}(\omega)/\text{Fin}$, and hence any such P can be embedded into (the quotient order of) \preceq , and therefore also into (the quotient order of) $\preceq_{\text{CT}}^\omega$.

Building on [LR05], in [FMR11] S.D. Friedman and the second author strengthened Theorem 1.1 by showing that the embeddability relation on countable models is (in the terminology of [CMMR13]) **invariantly universal**, i.e. that the following result holds.

THEOREM 1.4 (S.D. Friedman-Motto Ros). *For every Σ_1^1 quasi-order R there is an $\mathcal{L}_{\omega_1\omega}$ -sentence σ such that $R \sim_{\mathbf{B}} \preceq_\sigma^\omega$.*

Invariant universality is a strengthening of $\leq_{\mathbf{B}}$ -completeness, which just requires that every analytic quasi-order R is Borel reducible to $\preceq \upharpoonright \text{Mod}_{\mathcal{L}}^\omega$ (Theorem 1.1). Theorem 1.4 is obtained by applying the Lopez-Escobar theorem mentioned at the end of Section 1.1.1 to the following purely topological result — in fact we do not know if there is a direct proof of Theorem 1.4 which avoids going through Theorem 1.5.

THEOREM 1.5 (S.D. Friedman-Motto Ros). *For every Σ_1^1 quasi-order R there is a Borel $B \subseteq \text{Mod}_{\mathcal{L}}^\omega$ closed under isomorphism such that $R \sim_{\mathbf{B}} \preceq \upharpoonright B$.*

Various generalizations of Theorem 1.4 have already appeared in the literature: for example, in [FFH⁺12] the authors briefly consider its computable version, while [MR12] presents an extensive analysis of the possible interplay between the isomorphism and the embeddability relation on the same elementary class Mod_σ^ω for σ an $\mathcal{L}_{\omega_1\omega}$ -sentence.

1.2. What we wanted. The present paper was motivated by the quest for generalizations of Theorem 1.4 in two different directions:

- (A) considering quasi-orders belonging to more general pointclasses, such as the projective classes Σ_n^1 (for $n \geq 2$) and beyond, and
- (B) using more liberal (but still definable) kind of reductions.

Goal (A) is not an idle pursuit, since there are lots of equivalence relations and quasi-orders on Polish spaces naturally arising in mathematics which are not analytic. The following are a few examples of this sort.

EXAMPLE 1.6. Consider the quasi-order $(\mathcal{Q}, \leq_{\mathbf{B}})$ of Borel reducibility between analytic quasi-orders. As observed in [LR05], $\leq_{\mathbf{B}}$ is a Σ_3^1 relation in the codes for analytic quasi-orders, that is: there is a surjection $q: {}^\omega 2 \rightarrow \mathcal{Q}$ such that the relation

$$x \preceq_{\mathcal{Q}} y \Leftrightarrow q(x) \leq_{\mathbf{B}} q(y)$$

is Σ_3^1 . By [AK00], the restriction of $\preceq_{\mathcal{Q}}$ to (the codes for) countable Borel equivalence relations is already a *proper* Σ_2^1 relation.³

EXAMPLE 1.7. Let $X := (C([0; 1]))^\omega$ be the space of countable sequences of continuous, real-valued functions on the unit interval $[0; 1]$. Consider the following natural extension of the inclusion relation on X : given $\mathcal{F}, \mathcal{G} \in X$, we say that \mathcal{F} is *essentially contained* in \mathcal{G} (in symbols $\mathcal{F} \subseteq^{\text{lim}} \mathcal{G}$) if each function in \mathcal{F} can be obtained by recursively applying the pointwise limit operator to the functions in \mathcal{G} . This relation can be equivalently described as follows. Denote by $\text{lim}(\mathcal{F})$ the subset of $C([0; 1])$ generated by $\mathcal{F} \in X$ using pointwise limits, i.e. the smallest subset of $C([0; 1])$ containing \mathcal{F} and closed under the pointwise limit operation: then

$$\mathcal{F} \subseteq^{\text{lim}} \mathcal{G} \Leftrightarrow \text{lim}(\mathcal{F}) \subseteq \text{lim}(\mathcal{G}).$$

By a result of Becker (see e.g. [Kec95, p. 318]), it is easy to see that the relation \subseteq^{lim} is a *proper* Σ_2^1 quasi-order.

³To the best of our knowledge, the exact topological complexity of the full $\preceq_{\mathcal{Q}}$ is still unknown.

EXAMPLE 1.8. Consider the group $\text{Aut}(X)$ of all Borel automorphisms of a standard Borel space X , and the conjugacy relation on it. The elements of $\text{Aut}(X)$ can be coded as points of the Baire space, and the set \mathcal{A} of codes is $\mathbf{\Pi}_1^1$. Given $c \in \mathcal{A}$, let $f_c \in \text{Aut}(X)$ be the Borel automorphism coded by c . The equivalence relation $\{(c, d) \in \mathcal{A}^2 \mid f_c \text{ and } f_d \text{ are conjugated}\}$ is Σ_2^1 -complete [Cle07].

EXAMPLE 1.9. The density of a Borel set $A \subseteq {}^\omega 2$ at a point $x \in {}^\omega 2$ is the value $\mathcal{D}_A(x) := \lim_n \mu(A \cap \mathbf{N}_{x \upharpoonright n}^\omega) / \mu(\mathbf{N}_{x \upharpoonright n}^\omega)$, where μ is the usual measure on the Cantor space and $\mathbf{N}_{x \upharpoonright n}^\omega$ is the basic open neighborhood determined by $x \upharpoonright n$ (see Section 3). The limit might be undefined for some x , although by the Lebesgue density theorem $\mathcal{D}_A(x)$ is 0 or 1 for μ -almost every x . Thus $\mathcal{D}_A(x): {}^\omega 2 \rightarrow [0; 1]$ is a partial function and we may consider the quasi-order

$$A \preceq B \Leftrightarrow \text{ran}(\mathcal{D}_A) \subseteq \text{ran}(\mathcal{D}_B).$$

Since the density at a point x does not change if the set A is perturbed by a null set, the quasi-order \preceq makes sense on the measure algebra as well. It can be shown that \preceq is a $\mathbf{\Pi}_2^1$ -complete quasi-order on the measure algebra, and also on $\mathbf{K}({}^\omega 2)$, the hyperspace of all compact subsets of ${}^\omega 2$ (see [AC]).

Goal (B) is motivated by the desire of analyzing reducibilities for quasi-orders belonging to some well-behaved inner model. To illustrate the kind of situation we have in mind, let us mention the following two examples from the literature.

EXAMPLE 1.10. Consider reductions belonging to $L(\mathbb{R})$, the smallest inner model containing all the reals, and let $\leq_{L(\mathbb{R})}$ be the resulting reducibility relation between the quasi-orders of $L(\mathbb{R})$. Assume $\text{AD}^{L(\mathbb{R})}$, i.e. that all games with payoff set in $L(\mathbb{R})$ are determined — this assumption follows from e.g. strong forcing axioms (such as the Proper Forcing Axiom PFA), or from the existence of sufficiently large cardinals (infinitely many Woodin cardinals with a measurable above them suffice). Then many of the results on Borel reducibility between analytic binary relations can be extended to $L(\mathbb{R})$ by replacing $\leq_{\mathbf{B}}$ with $\leq_{L(\mathbb{R})}$, including e.g. the dichotomies of Silver and Glimm-Effros and the theory of turbulence — see [Hjo95, Hjo99, Hjo00].⁴ In this paper we shall repeatedly use the following remarkable result [Hjo00, Theorem 9.18].

THEOREM 1.11 (Hjorth). *If E_G is the orbit equivalence relation induced by a Polish group G acting in a turbulent way on a Polish space X (and hence E_G is Σ_1^1), then*

$$L(\mathbb{R}) \models E_G \not\leq_{L(\mathbb{R})} \cong \upharpoonright \text{Mod}_{\mathcal{L}}^\kappa$$

i.e. E_G is not reducible in $L(\mathbb{R})$ to the isomorphism relation \cong on the space of all \mathcal{L} -structures of size κ , for any countable language \mathcal{L} and any cardinal κ .

EXAMPLE 1.12. The Silver and the Glimm-Effros dichotomies have also been generalized to the ZFC-world by considering $\text{OD}(\mathbb{R})$ the inner model of real-ordinal definable sets in the Solovay's model (see [Ste84, Kan97, FK08]). In this framework, one compares quasi-orders in $\text{OD}(\mathbb{R})$ by means of the $\text{OD}(\mathbb{R})$ -reducibility $\leq_{\text{OD}(\mathbb{R})}$, namely the reducibility notion obtained by considering reductions in $\text{OD}(\mathbb{R})$.

REMARK 1.13. Even though both Examples 1.10 and 1.12 mainly concern generalizations of the same dichotomies for equivalence relations, the methods used to obtain them are quite different: in the former case the extensive knowledge of models of the Axiom of Determinacy AD is used, while in the latter forcing arguments together with absoluteness considerations are employed.

⁴As pointed out in [Hjo95], the generalization of Silver's dichotomy is due to Woodin.

Since both [LR05] and [FMR11] exploit the fact that a set is Σ_1^1 if and only if it is ω -Souslin, one can ask what sort of generalizations of Theorem 1.4 in the direction of goal (A) could be attained. Rather than looking at the more familiar projective classes Σ_n^1 mentioned in (A), as common sense would probably suggest, working with the pointclasses $\mathcal{S}(\kappa)$ of κ -Souslin sets for κ an uncountable cardinal (see Section 9) turns out to be the right move.⁵ This approach is not too restrictive, since e.g. under AD one has $\Sigma_n^1 = \mathcal{S}(\kappa_n)$ for an appropriate cardinal κ_n .

Remark 1.13 seems to suggest that if one aims at generalizing Theorem 1.4 to both the AC- and the AD-world, then different methods should be used. This prompts the question of which one of the two approaches is more promising for our purposes. On the one hand, the fact that we are going to study κ -Souslin quasi-orders seems to indicate that an AD-approach similar to the one of Example 1.10 should be preferred: in fact AD imposes an extremely rich structure on the subsets of Polish spaces and provides a well-developed general theory of κ -Souslin sets (for quite large cardinals κ), while we have very little information on the structure and properties of κ -Souslin sets in the context of AC, where the notion of κ -Souslin is nontrivial only when κ is smaller than the continuum 2^{\aleph_0} . On the other hand, any generalization of Theorem 1.4 to κ -Souslin quasi-orders seems to require replacing the space $\text{Mod}_{\mathcal{L}}^\omega$ with $\text{Mod}_{\mathcal{L}}^\kappa$, and the logic $\mathcal{L}_{\omega_1\omega}$ with $\mathcal{L}_{\kappa+\kappa}$. Unfortunately, there are two roadblocks down this path:

- a decent descriptive set theory on spaces like $\text{Mod}_{\mathcal{L}}^\kappa$, which can be identified with the generalized Cantor space ${}^\kappa 2$, seems to require cardinal arithmetic assumptions such as $|{}^{<\kappa}\kappa| = \kappa$ — see e.g. the generalization of the Lopez-Escobar theorem in Section 8.2, and [Vau75, MV93, FHK14] for more on these matters;
- the classical analysis of the logics $\mathcal{L}_{\kappa+\kappa}$ essentially requires the full Axiom of Choice AC.

Since both AC and $|{}^{<\kappa}\kappa| = \kappa > \omega$ contradict the Axiom of Determinacy, any generalization of Theorem 1.4 under AD seems out of reach. But even under AC, the cardinal condition $\kappa^{<\kappa} = \kappa > \omega$ cannot be achieved when $\kappa < 2^{\aleph_0}$, and the latter is needed to guarantee that the notion of κ -Souslinness is not trivial.

As there is a strong tension between the possible scenarios, any generalization of Theorem 1.4 may thus seem hopeless.

1.3. What we did. Despite the bleak outlook depicted in the previous section, we managed to prove some generalizations of the Louveau-Rosendal completeness result (Theorem 1.1) and of its strengthening to invariant universality (Theorem 1.4). Below is a collection of results that will be proved in Section 15. We would like to emphasize that they are not proved by *ad hoc* methods, but they all follow from the constructions and techniques of Section 10–13. This might serve as a partial justification for the length of this paper.

The central idea is that a quasi order R on a Polish space is reduced to the embeddability relation $\sqsubseteq_{\text{CT}}^\kappa$ on the collection CT_κ of all combinatorial trees of size κ . By Theorem 1.1, if R is Σ_1^1 , that is κ -Souslin with $\kappa = \omega$, then R is reducible to $\sqsubseteq_{\text{CT}}^\omega$; our results show that the higher the complexity of R , the larger the cardinal κ must be taken. This is reminiscent of a well-known feature in the proofs of determinacy where a game with a complex subset of the ${}^\omega\omega$ is transfigured in a closed game on ${}^\omega\kappa$ with κ large. Our reduction takes place between a Polish space and (a homeomorphic copy of) ${}^\kappa 2$, and the complexity of the reduction will be either $\kappa + 1$ -Borel or κ -Souslin-in-the-codes. We would like to stress that these notions are quite natural when working with ${}^\kappa 2$.

A miscellanea of results. For the sake of brevity, in the statements of Theorems 1.14 and 1.16 the quasi-order R is tacitly assumed to be defined on some Polish or standard Borel space, and the embeddability relation on Mod_σ^κ , the collection of models of size κ of the $\mathcal{L}_{\kappa+\kappa}$ -sentence σ , is

⁵The idea of considering κ -Souslin quasi-orders is not new: in fact, κ -Souslin (and co- κ -Souslin) quasi-orders on Polish spaces have been already extensively studied in the literature, see e.g. [HS82, She84, Kan97].

denoted by \leq_{σ}^{κ} . Recall also that the assumption $\text{AD}^{\text{L}(\mathbb{R})}$ used in some of the next results follows from sufficiently large cardinals or strong forcing axioms, such as PFA.

- THEOREM 1.14 (Completeness results). (a) Assume $\text{ZFC} + \text{PFA}$. Then $R \leq_{\Sigma_2^1} \leq_{\text{CT}}^{\omega_1}$ and $R \leq_{\mathbf{B}}^{\omega_1} \leq_{\text{CT}}^{\omega_1}$, for every Σ_2^1 quasi-order R . (Corollary 15.4)
- (b) Assume $\text{ZFC} + \forall x \in {}^\omega\omega (x^\# \text{ exists})$. Then $R \leq_{\mathbf{S}(\omega_2)} \leq_{\text{CT}}^{\omega_2}$ and $R \leq_{\mathbf{B}}^{\omega_2} \leq_{\text{CT}}^{\omega_2}$, for every Σ_3^1 quasi-order R . (Theorem 15.6)
- (c) Assume $\text{ZFC} + \text{AD}^{\text{L}(\mathbb{R})}$. Then $R \leq_{\mathbf{S}(\omega_{r(n)})} \leq_{\text{CT}}^{\omega_{r(n)}}$ and $R \leq_{\mathbf{B}}^{\omega_{r(n)}} \leq_{\text{CT}}^{\omega_{r(n)}}$, for every Σ_n^1 quasi-order R , where $r: \omega \rightarrow \omega$ is

$$(1.2) \quad r(n) = \begin{cases} 2^{k+1} - 2 & \text{if } n = 2k + 1, \\ 2^{k+1} - 1 & \text{if } n = 2k + 2. \end{cases}$$

(Theorem 15.8)

- (d) Assume $\text{ZF} + \text{DC} + \text{AD}$. For $n > 0$, let $\kappa_n := \lambda_n^1$ if n is odd and $\kappa_n := \delta_{n-1}^1$ if n is even.⁶ Then $R \leq_{\Sigma_n^1} \leq_{\text{CT}}^{\kappa_n}$ (and hence also $R \leq_{\mathbf{B}}^{\kappa_n} \leq_{\text{CT}}^{\kappa_n}$) for every Σ_n^1 quasi-order R . (Theorem 15.9)
- (e) Assume $\text{ZFC} + \text{AD}^{\text{L}(\mathbb{R})}$. Then $R \leq_{\text{L}(\mathbb{R})} \leq_{\text{CT}}^{\kappa}$ for every Σ_1^2 quasi-order R of $\text{L}(\mathbb{R})$, where $\kappa := \delta_1^2$ is defined as in Section 4.2. (Theorem 15.18)

REMARKS 1.15. (i) In Theorem 1.14, $\leq_{\mathbf{B}}^{\kappa}$ is the generalization of $\leq_{\mathbf{B}}$ to κ an uncountable cardinal (Definition 14.2). The reducibilities $\leq_{\Sigma_n^1}$ and $\leq_{\mathbf{S}(\kappa)}$ are instead the analogue of $\leq_{\mathbf{B}}$ where the reducing functions are required to be, respectively, Σ_n^1 -in-the-codes and $\mathbf{S}(\kappa)$ -in-the-codes (see Definition 5.3); with this notation, the standard Borel reducibility $\leq_{\mathbf{B}}$ would be denoted by $\leq_{\Sigma_1^1}$ and $\leq_{\mathbf{S}(\omega)}$, respectively. Finally, $\leq_{\text{L}(\mathbb{R})}$ is reducibility in $\text{L}(\mathbb{R})$ — see Example 1.10.

- (ii) The various statements in Theorem 1.14 can be seen as completeness results generalizing Theorem 1.1 to pointclasses Γ properly extending Σ_1^1 . The move to such Γ 's forces us to replace the Polish space CT_ω with CT_κ (for some $\kappa > \omega$), which is homeomorphic to (a closed subset of) the generalized Cantor space ${}^\kappa 2$, and hence far from being Polish. Therefore part (a) of Theorem 1.1 must necessarily be dropped in such generalizations (see also Remark 2.5).
- (iii) The reader familiar with Jackson's analysis of the regular cardinals below the projective cardinals in $\text{L}(\mathbb{R})$ will immediately see that parts (c) and (d) of Theorem 1.14 are strictly related.

The relations $\sim_{\mathbf{B}}^{\kappa}$ and $\sim_{\text{L}(\mathbb{R})}$ appearing in the next theorem are the bi-reducibility relations canonically associated to $\leq_{\mathbf{B}}^{\kappa}$ (see Remark 1.15(i)) and $\leq_{\text{L}(\mathbb{R})}$ (see Example 1.10), respectively.

- THEOREM 1.16 (Invariant universality results). (a) Assume $\text{ZF} + \text{AC}_\omega(\mathbb{R})$. Then for every Σ_2^1 quasi-order R there is an $\mathcal{L}_{\omega_2 \omega_1}$ -sentence σ such that $R \sim_{\mathbf{B}}^{\omega_1} \leq_{\sigma}^{\omega_1}$. (Theorem 15.3)
- (b) Assume $\text{ZFC} + \forall x \in {}^\omega\omega (x^\# \text{ exists})$. Then for every Σ_3^1 quasi-order R there is an $\mathcal{L}_{\omega_3 \omega_2}$ -sentence σ such that $R \sim_{\mathbf{B}}^{\omega_2} \leq_{\sigma}^{\omega_2}$. (Theorem 15.6)
- (c) Assume $\text{ZFC} + \text{AD}^{\text{L}(\mathbb{R})}$, and let $r: \omega \rightarrow \omega$ be as in equation (1.2). Then for every Σ_n^1 quasi-order R there is an $\mathcal{L}_{\omega_{r(n)+1} \omega_{r(n)}}$ -sentence σ such that $R \sim_{\mathbf{B}}^{\omega_{r(n)}} \leq_{\sigma}^{\omega_{r(n)}}$. (Theorem 15.8)
- (d) Assume $\text{ZF} + \text{DC} + \text{AD}$. Let $n \neq 0$ be an even number. Then for every Σ_n^1 quasi-order R there is an $\mathcal{L}_{\delta_n^1 \delta_{n-1}^1}$ -sentence⁷ σ such that $R \sim_{\mathbf{B}}^{\delta_{n-1}^1} \leq_{\sigma}^{\delta_{n-1}^1}$. (Theorem 15.9)

⁶See Section 4.2 for the relevant definitions. In particular, $\kappa_1 = \omega$, $\kappa_2 = \omega_1$, $\kappa_3 = \aleph_\omega$, and $\kappa_4 = \aleph_{\omega+1}$.

⁷See Section 4.2 for the definition of the projective ordinals δ_n^1 , and recall that when n is even we have $\delta_n^1 = (\delta_{n-1}^1)^+$ (under the assumption $\text{ZF} + \text{DC} + \text{AD}$).

- (e) Assume $\text{ZFC} + \text{AD}^{L(\mathbb{R})}$. Then for every Σ_1^2 quasi-order R of $L(\mathbb{R})$ there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ belonging to $L(\mathbb{R})$ such that $R \sim_{L(\mathbb{R})} \sqsubset_{\sigma}^{\kappa}$, where as in Theorem 1.14(e) we set $\kappa := \delta_1^2$. (Theorem 15.18)

By considering the equivalence relation associated to a quasi-order, the results above can be turned into statements concerning equivalence relations on Polish or standard Borel spaces and the bi-embeddability relation $\approx_{\text{CT}}^{\kappa}$ on combinatorial trees of size κ . The stark difference between bi-embeddability \approx and isomorphism \cong on countable models uncovered by Theorem 1.1 (see the observation after Remark 1.2) is also present in the uncountable case: assuming $\text{AD}^{L(\mathbb{R})}$, by Theorem 1.14(e) any Σ_1^2 equivalence relation of $L(\mathbb{R})$ is $\leq_{L(\mathbb{R})}$ -reducible to $\approx_{\mathcal{L}}^{\delta_1^2}$, the bi-embeddability relation on $\text{Mod}_{\mathcal{L}}^{\delta_1^2}$, while this badly fails if \approx is replaced by \cong , by Hjorth's Theorem 1.11.

By applying Theorems 1.14 and 1.16 we also get some information on the Σ_3^1 quasi-order $(\mathcal{Q}, \leq_{\mathbf{B}})$ of Example 1.6. Such quasi-order may be seen as a (definable) embeddability relation between structures of size the continuum. It turns out that, under suitable hypotheses, $(\mathcal{Q}, \leq_{\mathbf{B}})$ can be turned into an embeddability relation between *well-ordered* structures of size potentially smaller than the continuum. Indeed, in models with choice we have:

THEOREM 15.7. Assume $\text{ZFC} + \forall x \in {}^\omega\omega (x^\# \text{ exists})$. Then the quotient order of $(\mathcal{Q}, \leq_{\mathbf{B}})$ (definably) embeds into the quotient order of $\sqsubset_{\text{CT}}^{\aleph_2}$. Moreover, there is an $\mathcal{L}_{\aleph_3 \aleph_2}$ -sentence σ such that the quotient orders of $(\mathcal{Q}, \leq_{\mathbf{B}})$ and $\sqsubset_{\sigma}^{\aleph_2}$ are (definably) isomorphic.

On the other hand, in the determinacy world we get:

THEOREM 15.11. Assume $\text{ZF} + \text{DC} + \text{AD}$. Then the quotient order of $(\mathcal{Q}, \leq_{\mathbf{B}})$ (definably) embeds into the quotient order of $\sqsubset_{\text{CT}}^{\aleph_\omega}$. Moreover, there is an $\mathcal{L}_{\aleph_{\omega+1} \aleph_\omega}$ -sentence σ such that the quotient orders of $(\mathcal{Q}, \leq_{\mathbf{B}})$ and $\sqsubset_{\sigma}^{\aleph_\omega}$ are (definably) isomorphic.

In Section 16.1 we generalize the combinatorial completeness property of $\sqsubset_{\text{CT}}^{\omega}$ described in Example 1.3 to the uncountable case, albeit in a weaker form, in both the AC- and the AD-world.

PROPOSITION 16.1. Assume ZFC and let $\omega < \kappa \leq 2^{\aleph_0}$. Then every partial order P of size κ can be embedded into the quotient order of $\sqsubset_{\text{CT}}^{\kappa}$. In fact, for every such P there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ (all of whose models are combinatorial trees) such that the quotient order of $\sqsubset_{\sigma}^{\kappa}$ is isomorphic to P .

PROPOSITION 16.2. Assume ZFC and let $\omega < \kappa \leq 2^{\aleph_0}$. Then $\subseteq_{\kappa}^* \leq_{\mathbf{B}}^{\kappa} \sqsubset_{\text{CT}}^{\kappa}$, where \subseteq_{κ}^* is the relation on $\mathcal{P}(\kappa)$ of inclusion modulo bounded subsets. In particular, every linear order of size \aleph_{n+1} can be embedded into the quotient order of $\sqsubset_{\text{CT}}^{\aleph_n}$ (whenever $2^{\aleph_0} \geq \aleph_n$).

THEOREM 16.4. Assume $\text{ZF} + \text{DC} + \text{AD}$. Let κ be a Souslin cardinal. Then every partial order P of size κ can be embedded into the quotient order of $\sqsubset_{\text{CT}}^{\kappa}$. In fact, if $\kappa < \delta_{\mathcal{S}(\kappa)}^1$, then for every such P there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ (all of whose models are combinatorial trees) such that the quotient order of $\sqsubset_{\sigma}^{\kappa}$ is isomorphic to P .

Notice that by Proposition 9.25(c) we can apply the second part of Theorem 16.4 to any Souslin cardinal κ (in a model of $\text{ZF} + \text{AD} + \text{DC}$) which is not a regular limit of Souslin cardinals: in particular, we can take κ to be one of the projective ordinals δ_n^1 .

Finally, in Section 16.2 we show that in all our completeness results (including the ones mentioned so far in this introduction) one may freely replace embeddings and combinatorial trees with other kinds of morphisms and structures which are relevant to graph theory and model theory, namely:

- we can consider full homomorphisms between graphs (or even just combinatorial trees);

- we may also consider embeddings between (complete) lattices, where the latter may indifferently be construed as partial orders or as bounded lattices in the algebraic sense.

1.4. How we proved it. In order to overcome the difficulties explained at the end of Section 1.2, we forwent the approach followed in the countable case: this would have required to first analyze the descriptive set theory of $\text{Mod}_{\mathcal{L}}^\kappa$ in order to achieve a generalization of Theorem 1.5, and then obtain as a corollary the corresponding generalization of Theorem 1.4. The key idea is to reverse this approach and to exploit the greater expressive power of the logic $\mathcal{L}_{\kappa+\kappa}$ when κ is uncountable and directly extend Theorem 1.4 as follows: to each κ -Souslin quasi-order R on the Cantor space ${}^\omega 2$ we associate an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ (all of whose models are combinatorial trees) so that R is bi-reducible to \sqsubset_σ^κ , in symbols $R \sim \sqsubset_\sigma^\kappa$ (Theorem 12.15). The corresponding topological version is obtained as a corollary (using a restricted version of the generalized Lopez-Escobar theorem) by using formulæ belonging to a sufficiently powerful fragment $\mathcal{L}_{\kappa+\kappa}^b$ of $\mathcal{L}_{\kappa+\kappa}$ (b is for *bounded*). To be more precise: given a tree T on $2 \times 2 \times \kappa$ such that $R = p[T]$ is a quasi-order, we shall construct in ZF

- a function $f_T: {}^\omega 2 \rightarrow \text{CT}_\kappa$ (see (11.5)) such that f_T reduces the quasi-order R to $\sqsubset_{\text{CT}}^\kappa$ and satisfies $f_T(x) \cong f_T(y) \Leftrightarrow x = y$ (Theorem 11.8);
- an $\mathcal{L}_{\kappa+\kappa}$ -sentence $\sigma = \sigma_T$ and a function $h_T: \text{Mod}_\sigma^\kappa \rightarrow {}^\omega 2$ such that Mod_σ^κ is the closure under isomorphism of the range of f_T , and h_T reduces $\sqsubset_{\text{CT}}^\kappa$ to R (Corollary 12.14).

This construction depends only on the cardinal $\kappa > \omega$ and on the chosen witness T of the fact that R is κ -Souslin. This means that the desired $\mathcal{L}_{\kappa+\kappa}$ -sentence σ and the two reductions f_T, h_T witnessing $R \sim \sqsubset_\sigma^\kappa$ can be found in every inner model containing κ and T , and that they are in fact (essentially) the same in all these models. Moreover, the reductions involved are absolute, as they can be defined by formulæ (in the language of set theory) which define reductions between $R = p[T]$ and \sqsubset_σ^κ in every generic extension and in every inner model of the universe of sets V we started with. Such reductions have essentially the same topological complexity of the quasi-order R : for example, if κ is regular then they are $\kappa+1$ -Borel, a natural extension of the classical notion of a Borel function (see Definition 5.1). Therefore our results provide natural generalizations of both Theorems 1.4 and 1.5 (which correspond to the basic case $\kappa = \omega$) to uncountable κ 's — see Theorem 14.8 and 14.10, respectively. All these observations lead us to consider also the more general notion of *definable cardinality*, which is strictly related to the notion of *definable reducibility* — see Section 14 for a more thorough discussion on the genesis and relevance of these concepts.

It is worth pointing out that in order to find applications in both the AC-world and the AD-world, the above mentioned preliminary completeness and invariant universality results for κ -Souslin quasi-orders (for κ an uncountable cardinal) *are developed in ZF*, so that they can be applied to *all* situations in which $\mathcal{S}(\kappa)$ is a nontrivial pointclass. These include, among many others, the cases of the projective pointclasses Σ_n^1 , of the Σ_1^2 sets of $L(\mathbb{R})$, and even of the entire $\mathcal{P}(\mathbb{R})$ (under various determinacy or large cardinal assumptions).

Let us conclude this section with a more general comment. There is a common phenomenon in model theory: the study of uncountable models requires ideas and techniques quite different from the ones used in the study of countable models, and this paper is no exception. As already observed in Section 1.2 and at the beginning of this subsection, the techniques used in [FMR11] for the countable case cannot be transferred to the uncountable case. Conversely, the arguments contained in this paper cannot be adapted to include the countable case, since we essentially need to be able to use a single $\mathcal{L}_{\kappa+\kappa}$ -sentence either to assert the existence of certain *infinite* (but still small) substructures (see Section 12), or to express the *well-foundedness* of a binary relation (see Section 13). As it is well-known, none of these possibilities can be achieved using the logic $\mathcal{L}_{\omega_1\omega}$.

1.5. Classification of non-separable structures up to bi-embeddability. As recalled at the end of Section 1.1.1, the problem of classifying e.g. Polish metric spaces up to isometry or separable Banach spaces up to linear isometry have been widely studied in the literature, but very little was known about the analogous classification problems up to bi-embeddability. In [LR05] Louveau and Rosendal used their Theorem 1.1 to show that many kinds of separable spaces are in fact not classifiable (in any reasonable way) with respect to the relevant notion of bi-embeddability. This is made precise by the following result. (Notice that the relations appearing in the next theorem can all be construed as analytic quasi-orders on corresponding standard Borel spaces — see [LR05] for more details.)

THEOREM 1.17 (Louveau-Rosendal). *The following relations are $\leq_{\mathbf{B}}$ -complete analytic quasi-orders:*

- (a) *continuous embeddability between compacta ([LR05, Theorem 4.5]);*
- (b) *isometric embeddability between (ultrametric or discrete) Polish metric spaces ([LR05, Propositions 4.1 and 4.2]);*
- (c) *linear isometric embeddability between separable Banach spaces ([LR05, Theorem 4.6]).*

As a consequence, the corresponding bi-embeddability relations are $\leq_{\mathbf{B}}$ -complete analytic equivalence relations.

(Theorem 1.17 has been further improved in [CMMR13], where it is shown that it is possible to obtain analogues of Theorem 1.5 in which $\sqsubset_{\mathbf{CT}}^\omega$ is replaced by e.g. any of the embeddability relations mentioned in Theorem 1.17.)

Following this line of research, in Sections 16.3 and 16.4 we study the complexity of various classification problems for non-separable spaces. In particular, we show that in all our completeness results one may systematically replace the embeddability relation between combinatorial trees of size κ with e.g. the isometric embeddability relation between (discrete and/or ultrametric) complete metric spaces of density character κ . Here is a small sample of the results that may be obtained in this way (see also Theorem 16.22 and Remark 16.24 for more results of this kind). In the next theorem, \sqsubseteq^i denotes the relation of isometric embeddability between metric spaces.

- THEOREM 1.18.** (a) *Assume ZFC. Then the relation \sqsubseteq^i between (discrete) complete (ultra)metric spaces of density character ω_1 is $\leq_{\mathbf{B}}^{\omega_1}$ -complete for Σ_2^1 quasi-orders on standard Borel spaces. If moreover we assume either $\text{AD}^{\text{L}(\mathbb{R})}$ or $\text{MA} + \neg\text{CH} + \exists a \in {}^\omega\omega (\omega_1^{\text{L}[a]} = \omega_1)$, then such relation is also $\leq_{\Sigma_2^1}$ -complete for Σ_2^1 quasi-orders on standard Borel spaces.*
- (b) *Assume $\text{ZFC} + \forall x \in {}^\omega\omega (x^\# \text{ exists})$. The relation \sqsubseteq^i between (discrete) complete (ultra)metric spaces of density character ω_2 is $\leq_{\mathbf{B}}^{\omega_2}$ -complete for Σ_3^1 quasi-orders on standard Borel spaces.*
- (c) *Assume $\text{ZFC} + \text{AD}^{\text{L}(\mathbb{R})}$. Then the relation \sqsubseteq^i between (discrete) complete (ultra)metric spaces of density character $\omega_{r(n)}$ is $\leq_{\mathbf{B}}^{\omega_{r(n)}}$ -complete for Σ_n^1 quasi-orders on standard Borel spaces, where r is as in equation (1.2).*
- (d) *Assume $\text{ZF} + \text{DC} + \text{AD}$. For $0 \neq n \in \omega$, let κ_n be as in Theorem 1.14(d). The relation \sqsubseteq^i between (discrete) complete (ultra)metric spaces of density character κ_n is both $\leq_{\mathbf{B}}^{\kappa_n}$ -complete and $\leq_{\Sigma_n^1}$ -complete for Σ_n^1 quasi-orders on standard Borel spaces.*

In particular, it follows from Theorem 1.18 that the problem of classifying non-separable complete metric spaces up to isometric bi-embeddability is extremely complicated. We show that for uncountable κ 's, both the isometry relation and the isometric bi-embeddability relation between (discrete) complete metric spaces of density character κ may consistently have maximal complexity with respect to the relevant reducibility notion $\leq_{\mathbf{B}}^\kappa$ (Corollary 16.16). We also show that in models of $\text{AD}^{\text{L}(\mathbb{R})}$ the relation of isometric bi-embeddability between ultrametric or discrete complete metric spaces of a given uncountable density character is way more complicated

(with respect to $L(\mathbb{R})$ -reducibility) than the isometry relation on the same class: the former $\leq_{L(\mathbb{R})}$ -reduces, among others, all Σ_2^1 equivalence relations on a Polish or standard Borel space, while the latter cannot even $\leq_{L(\mathbb{R})}$ -reduce all Σ_1^1 equivalence relations (see the comment after Theorem 16.26). These observations show in particular that it is independent of $\mathbf{ZF} + \mathbf{DC}$ whether e.g. the relation of isometric bi-embeddability between ultrametric (respectively, discrete) complete metric spaces of density character ω_2 is $\leq_{\mathbf{B}}^{\omega_2}$ -reducible to the isometry relation on the same class of spaces (Corollary 16.27).

Finally, in Section 16.4 we show that in all previously mentioned results (including e.g. Theorem 1.18) one can further replace the isometric embeddability relation \sqsubseteq^i between complete metric spaces of density character κ with the linear isometric embeddability relation \sqsubseteq^{li} between Banach spaces of density κ . Thus also the problem of classifying non-separable Banach spaces up to linear isometry or linear isometric bi-embeddability is very complicated — in fact these equivalence relations may consistently have maximal complexity as well (Corollary 16.32).

In our opinion, these (anti-)classification results for non-separable complete metric spaces and Banach spaces constitute, together with the tight connections between $\kappa+1$ -Borel reducibility $\leq_{\mathbf{B}}^{\kappa}$ and Shelah’s stability theory uncovered in [FHK14], some of the strongest motivations for pursuing the research in the currently fast-growing field of generalized descriptive set theory.

1.6. Organization of the paper, or: How (not) to read this paper. The aspiration of this paper, besides that of presenting new results, is to serve as a basic reference text for works dealing with generalized descriptive set theory, and at the same time to be as self-contained as possible. For this reason, in Sections 2–9 we collected all relevant definitions and surveyed all basic results involved in the rest of the work. These sections contain many well-known folklore facts or elementary observations which unfortunately, to the best of our knowledge, cannot be found in a unitary and organic presentation elsewhere, together with some new results which may be of independent interest.

Due to the applications we had in mind, we developed these preliminaries trying to minimize the amount of set-theoretic assumptions or cardinal conditions required to prove them: this often forced us to find new proofs and ideas which, albeit slightly more involved than the “classical” ones, allowed us to work in wider or less standard contexts. As a byproduct, we obtained independence results concerning the generalized Cantor space ${}^{\omega_1}2$ which are invisible when working under $\mathbf{ZFC} + \mathbf{CH}$, the most popular setup in the current literature to deal with such space (see e.g. Remark 3.13(iii) or the comment after Proposition 6.7).

For the reader’s convenience, Sections 3–5 are organized as follows: all the definitions and statements crucial for the rest of the paper are collected in its first subsection, while their proofs, a more detailed and thorough discussion on the subject under consideration, together with additional information and results, are postponed to the later subsections, which are marked with a *. *Readers only interested in the results on the embeddability relation and in their applications, may safely skip these starred subsections.*

The authors are aware of Bertrand Russell’s aphorism: “A book should have either intelligibility or correctness: to combine the two is impossible, ...”. Although we tried our best to avoid writing incorrect statements, whenever a choice had to be made between intelligibility and correctness, we opted for the former. We are also very intimidated by the second part of such aphorism: “...but to lack both is to be unworthy of such a place as Euclid has occupied in education”, hoping we did not fail so badly in this endeavor.

1.7. Annotated content. Here is a synopsis of the sections of the paper and their content.

Section 2: Preliminaries and notation. Here we collect all the basic notions and facts that are taken for granted in this paper.

Section 3: The generalized Cantor space. We study the basic properties of the generalized Cantor space ${}^\kappa 2$ endowed with various topologies. We consider both the *bounded topology* τ_b and the *product topology* τ_p , as well as many other intermediate topologies. We also introduce in this context the notions of Lipschitz and continuous reducibility together with the corresponding (long) reduction games, and use game theoretic arguments to prove (without any determinacy assumption) some technical properties that are used in the later sections.

Section 4: Generalized Borel sets. We introduce the collection of α -Borel subsets (and their effective counterpart) of a given topological space X as a generalization to higher cardinals of the classical notion of a Borel set. The notion of α -Borel subset of \mathbb{R} has been thoroughly studied under determinacy assumptions in the Seventies-Eighties (see Cabal volumes), and the notion of α -Borelness also turns out to be the right generalization of Borelness when $X = {}^\kappa 2$ for κ an uncountable cardinal. Using the results on Lipschitz and Wadge reducibility from the previous section, we show in particular that under ZFC the $\kappa + 1$ -Borel hierarchy on ${}^\kappa 2$ does not collapse for every infinite cardinal κ .⁸

Section 5: Generalized Borel functions. We study $\kappa + 1$ -Borel functions and Γ -in-the-codes functions between generalized Cantor spaces. These are natural generalizations in two different directions of the notion of a Borel function between Polish spaces, which would correspond to the cases $\kappa = \omega$ and $\Gamma = \Sigma_1^1$ (equivalently, $\Gamma = \mathcal{S}(\omega)$), respectively.

Section 6: The generalized Baire space and Baire category. We consider the generalized Baire space ${}^\kappa \kappa$ and some of its relevant subspaces, including e.g. the group $\text{Sym}(\kappa)$ of permutations of κ , and extend to them the analysis from Section 3. We determine under which conditions ${}^\kappa \kappa$ and ${}^\kappa 2$ are homeomorphic, notably including the cases when κ is singular or when we are working in models of determinacy (such cases have not been considered in the literature so far). Finally, we prove some Baire category results which, as usual, can be recast in terms of forcing axioms.

Section 7: Standard Borel κ -spaces, κ -analytic quasi-orders, and spaces of codes. Working in ZF, we introduce a very general notion of standard Borel κ -space (which in the classical setting $\text{ZFC} + \kappa^{<\kappa} = \kappa$ coincides with the one introduced in [MR13]), and we consider κ -analytic relations on such spaces. We also introduce various nice spaces of codes for uncountable structures or non-separable spaces, including the space $\text{Mod}_{\mathcal{L}}^\kappa$ of (codes for) \mathcal{L} -structures of size κ , the space \mathfrak{M}_κ of (codes for) complete metric spaces of density character κ , and the space \mathfrak{B}_κ of (codes for) Banach spaces of density κ . These spaces of codes are all standard Borel κ -spaces, and the corresponding isomorphism/embeddability relations on them are κ -analytic equivalence relations/quasi-orders.

Section 8: Infinitary logics and models. We recall the model theoretic notions that are used in the sequel, including the infinitary logics $\mathcal{L}_{\kappa+\kappa}$, and we prove various generalizations to uncountable structures of the Lopez-Escobar theorem mentioned at the end of Section 1.1.1 (some of these generalizations were independently obtained also in [FHK14]). Although a full generalization can be obtained in ZFC only assuming $\kappa^{<\kappa} = \kappa$, a careful analysis of the proof leads to several intermediate results. We also introduce the bounded logic $\mathcal{L}_{\kappa+\kappa}^b$, a powerful enough fragment of $\mathcal{L}_{\kappa+\kappa}$ which avoids the pitfalls of naive generalizations of the Lopez-Escobar theorem. This is crucial for many of our main results.

⁸This is an illuminating example of how new and more refined arguments allow us to drop unnecessary assumptions on the cardinal κ . In the literature this result is usually proved (with a straightforward generalization of the classical argument using universal sets and diagonalization) only for cardinals κ satisfying $\kappa^{<\kappa} = \kappa$. In contrast, our proof allows us to also deal with cardinals satisfying $\kappa^{<\kappa} > \kappa$ (including the singular ones), where the required universal sets do not exist at all.

Section 9: κ -Souslin sets. We briefly study the pointclass $\mathcal{S}(\kappa)$ of κ -Souslin subsets of Polish spaces and the collection of Souslin cardinals. These notions have been extensively studied under the assumption $\text{ZF} + \text{DC} + \text{AD}$: we both review those results from this beautiful and deep area of research which are relevant to our work, and prove analogous results (when possible) under the assumption ZFC . In particular, we compute the exact value of the cardinal $\delta_{\mathcal{S}(\kappa)}$ associated to the pointclass $\mathcal{S}(\kappa)$. To the best of our knowledge this computation has been overlooked in the literature (even in the AD -context), so this result might be of independent interest.

Section 10: The main construction. In this section we show how to construct certain combinatorial trees (i.e. acyclic connected graphs) of size κ starting from a descriptive set-theoretic tree T on $2 \times \kappa$: this is the main technical construction that gives our completeness and invariant universality results.

Section 11: Completeness. Using the construction from the previous section, we show that the relation $\sqsubseteq_{\text{CT}}^\kappa$ of embeddability between combinatorial trees of size κ is complete for the class of κ -Souslin quasi-orders on Polish or standard Borel spaces.

Section 12: Invariant universality. We improve the result from Section 11 by showing that $\sqsubseteq_{\text{CT}}^\kappa$ is indeed invariantly universal for the same class of quasi-orders.

Section 13: An alternative approach. We provide a modification of the main construction and results from the previous three sections: this variant yields a generalization of Theorem 1.5 that cannot be achieved using the construction and results from Sections 10–12.

Section 14: Definable cardinality and reducibility. We discuss various forms of definable cardinality and reducibility that have already been considered in the literature, and translate our main results in corresponding statements involving these concepts.

Section 15: Some applications. We present a selection of applications of the results from Section 14 in some of the most important and well-known frameworks, leading to those natural results which, as noticed at the beginning of this introduction, were the original motivation for the entire work and for the technical developments contained in it.

Section 16: Further completeness results. We show that $\sqsubseteq_{\text{CT}}^\kappa$ is complicated also from the purely combinatorial point of view by embedding in it various partial orders and quasi-orders. Moreover, we provide some model theoretic variants of our completeness results concerning full homomorphisms between graphs and embeddings between lattices. Finally, we prove some results on the complexity of the relations of isometry and isometric (bi-)embeddability between complete metric spaces of uncountable density character, as well as on the complexity of the relations of linear isometry and linear isometric (bi-)embeddability between non-separable Banach spaces.

2. Preliminaries and notation

2.1. Basic notions. Our notation is standard as in most text in set theory, such as [Mos09, Kec95, Kan03, Jec03]. By ZF we mean the Zermelo-Fr enkel set theory, which is formulated in the language of set theory LST , i.e. the first-order language with \in as the only non-logical symbol; \subseteq means subset and \subset means *proper* subset, $\mathcal{P}(X)$ is the powerset of X , the disjoint union and the symmetric difference of X and Y are denoted by $X \uplus Y$ and $X \triangle Y$ respectively, and so on. For the reader's convenience we collect here all definitions and basic facts that are used later.

2.1.1. *Ordinals and cardinals.* Ordinals are denoted by lower case Greek letters, and Ord is the class of all ordinals. A **cardinal** is an ordinal not in bijection with a smaller ordinal. The class of all infinite cardinals is Card , and the letters κ, λ, μ always denote an element of Card . For $\alpha \geq \omega$ we denote by α^+ the least cardinal larger than α . The **cofinality** of κ is the smallest cardinal λ such that there is a cofinal $j: \lambda \rightarrow \kappa$, i.e. $\forall \alpha < \kappa \exists \beta < \lambda (\alpha \leq j(\beta))$. A cardinal κ is called **regular** if $\text{cof}(\kappa) = \kappa$ and **singular** otherwise.

We use

$$(2.1) \quad \langle \cdot, \cdot \rangle: \text{Ord} \times \text{Ord} \rightarrow \text{Ord},$$

for the inverse of the enumerating function of the well-order \preceq on $\text{Ord} \times \text{Ord}$ defined by

$$(\alpha, \beta) \preceq (\gamma, \delta) \Leftrightarrow \max\{\alpha, \beta\} < \max\{\gamma, \delta\} \vee [\max\{\alpha, \beta\} = \max\{\gamma, \delta\} \wedge (\alpha, \beta) \leq_{\text{lex}} (\gamma, \delta)],$$

where \leq_{lex} is the lexicographic ordering. This pairing function maps $\kappa \times \kappa$ onto κ , for $\kappa \in \text{Card}$. Let

$$(2.2) \quad \langle\langle \cdot \rangle\rangle: {}^{<\omega}\text{Ord} \rightarrow \text{Ord}$$

be a bijection that maps ${}^{<\omega}\kappa$ onto κ , for $\kappa \in \text{Card}$ — see e.g. [Kun80, Lemma 2.5, p. 159].

2.1.2. *Functions and sequences.* Unless otherwise specified, all functions $f: X \rightarrow Y$ are assumed to be total. We write $f: X \rightarrowtail Y$ and $f: X \twoheadrightarrow Y$ to mean that f is injective and that f is surjective, respectively. Similarly, the symbols $X \rightarrowtail Y$ and $X \twoheadrightarrow Y$ means that there is an injection of X into Y and that Y is a surjective image of X , respectively. The pointwise image of $A \subseteq X$ via $f: X \rightarrow Y$ is $\{f(a) \mid a \in A\}$ and it is denoted either by $f(A)$ or by $f''A$ — the first notation follows the tradition in analysis and descriptive set theory, the second is common in set theory especially when A is transitive.⁹ The set of all functions from X to Y is denoted by XY , while

$$(2.3) \quad {}^X(Y) := \{f \in {}^XY \mid f \text{ is injective}\}.$$

For κ an infinite cardinal let

$$\mathbf{Fn}(X, Y; \kappa) := \{s \mid s: u \rightarrow Y \wedge u \subseteq X \wedge |u| < \kappa\}.$$

Let ${}^{<\alpha}Y := \bigcup_{\gamma < \alpha} {}^\gamma Y$, and let

$$\mathbf{Fn}(\kappa, Y; b) := {}^{<\kappa}Y$$

be the set of all sequences with values in Y and length $< \kappa$. For $\alpha, \beta \in \text{Ord}$ let

$$[\alpha]^\beta := \{u \subseteq \alpha \mid \text{ot}(u) = \beta\}.$$

An element u of $[\alpha]^\beta$ is identified with its (increasing) enumerating function $u: \beta \rightarrow \alpha$. The sets $\leq^\alpha X$, $[\alpha]^{<\beta}$, and $[\alpha]^{\leq\beta}$ are defined similarly. The length of $u \in {}^{<\alpha}X$ is denoted by $\text{lh } u$. The concatenation of $u \in {}^{<\omega}X$ with $v \in {}^{\leq\omega}X$ is denoted by $u \hat{\ } v$. When dealing with sequences of length 1 we shall often blur the distinction between the element and the sequence, and write e.g. $u \hat{\ } x$ rather than $u \hat{\ } \langle x \rangle$. For $x \in X$ and $\alpha \in \text{Ord}$, the sequence of length α constantly equal to x is denoted by $x^{(\alpha)}$. If $\emptyset \neq u \in {}^{<\omega}X$ then

$$(2.4) \quad u^* := u \restriction \text{lh}(u) - 1$$

is the finite sequence obtained by deleting the last element from u .

⁹The notation $f[A]$ for the pointwise image is also common in the literature, but we eschew it as square brackets are already used for equivalence classes and for the body of trees — see Section 2.6.2.

2.2. Choice and determinacy. Since several results in this paper concern models of set theory where the **Axiom of Choice** (AC) may or may not hold, we shall state explicitly any assumption used beyond the axioms of ZF. Among such assumptions are the **Axiom of Countable Choices** (AC_ω), the **Axiom of Dependent Choices** (DC), their versions restricted to the reals $\text{AC}_\omega(\mathbb{R})$ and $\text{DC}(\mathbb{R})$, and the so-called determinacy axioms, namely AD and its stronger version $\text{AD}_\mathbb{R}$. The **Axiom of Determinacy** AD is the statement that for each $A \subseteq {}^\omega\omega$ the zero-sum, perfect information two-players game G_A^ω is **determined**. That is to say that one of the two players **I** and **II** has a winning strategy in G_A^ω where they take turns in playing $n_0, n_1, \dots \in \omega$ and **I** wins if and only if $\langle n_i \mid i \in \omega \rangle \in A$. The **Axiom of Real Determinacy** $\text{AD}_\mathbb{R}$ asserts that every $G_A^\mathbb{R}$ is determined, where $A \subseteq {}^\omega\mathbb{R}$ and **I** and **II** play $x_0, x_1, \dots \in \mathbb{R}$. The principle AD implies many regularity properties, such as: every set of reals is Lebesgue measurable, has the property of Baire, and has the **Perfect Set Property** (PSP), i.e. either it is countable, or it contains a homeomorphic copy of ${}^\omega 2$.

We also occasionally consider the **Axiom of κ -Choices** (AC_κ), asserting that the product of κ -many nonempty sets is nonempty, its restriction to the reals $\text{AC}_\kappa(\mathbb{R})$, the **Continuum Hypothesis** (CH), and various **forcing axioms** like MA_{ω_1} , PFA, and so on.

Although AD and $\text{AD}_\mathbb{R}$ forbid any well-ordering of the reals, and therefore are incompatible¹⁰ with AC, they are consistent with DC. In fact they imply weak forms of choice. For example, $\text{AD}_\mathbb{R}$ implies the **Uniformization Property** (Unif): every function $f: \mathbb{R} \rightarrow \mathbb{R}$ has a right inverse, i.e. a function $g: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(g(x)) = x$ for all $x \in \mathbb{R}$; equivalently, for every set $A \subseteq \mathbb{R}^2$ there is a function f with domain $\{x \in \mathbb{R} \mid \exists y \in \mathbb{R} ((x, y) \in A)\}$ such that $(x, f(x)) \in A$. Uniformization does not follow from AD: in $\text{L}(\mathbb{R})$, the canonical inner model for AD, Σ_1^2 is the largest collection of sets that can be uniformized, and by an unpublished theorem of Woodin's, $\text{AD}_\mathbb{R}$ and $\text{AD} + \text{Unif}$ are equivalent over $\text{ZF} + \text{DC}$.

2.3. Cardinality. We write $X \asymp Y$ to say that X and Y are in bijection, and $[X]_\asymp$ is the collection of all sets Y of minimal rank that are in bijection with X , i.e. it is the equivalence class of X under \asymp cut-down using Scott's trick. Working in ZF, the **cardinality** of a set X is defined to be

$$|X| := \begin{cases} \text{the unique cardinal } \kappa \asymp X & \text{if } X \text{ is well-orderable,} \\ [X]_\asymp & \text{otherwise.} \end{cases}$$

Thus AC implies that every cardinality is a cardinal. Set $|X| \leq |Y|$ if and only if $X \hookrightarrow Y$. By the Schröder-Bernstein theorem, $|X| = |Y|$ if and only if $|X| \leq |Y| \leq |X|$.

When λ is a cardinal κ^λ denotes cardinal exponentiation, i.e. the cardinality of ${}^\lambda\kappa$, while $\kappa^{<\lambda} := |{}^{<\lambda}\kappa| = \sup \{\kappa^\nu \mid \nu < \lambda \wedge \nu \text{ a cardinal}\}$. Whenever we write κ^λ (for λ and infinite cardinal) or $\kappa^{<\lambda}$ (for λ an *uncountable* cardinal) the Axiom of Choice AC is tacitly assumed. Given a cardinal κ and a set X , we let $\mathcal{P}_\kappa(X) := \{Y \subseteq X \mid |Y| < \kappa\}$.

The next result is straightforward under choice — the main reason to explicitly give a proof here, is to show that it is provable in ZF.

PROPOSITION 2.1. *Suppose X has at least two elements and λ is an infinite regular cardinal. Then*

$$|{}^{<\lambda}({}^{<\lambda}X)| = |\lambda \times {}^{<\lambda}X| = |{}^{<\lambda}X|.$$

PROOF. Fix distinct $x_0, x_1 \in X$, and for $s \in {}^{<\lambda}X$ let $s' := x_0^{(\text{lh } s)} \hat{\ } x_1 \hat{\ } s$. Then

$${}^{<\lambda}({}^{<\lambda}X) \rightarrow {}^{<\lambda}X, \quad \vec{s} = \langle s_\beta \mid \beta < \text{lh } \vec{s} \rangle \mapsto s'_0 \hat{\ } s'_1 \hat{\ } \dots \hat{\ } s'_\beta \hat{\ } \dots,$$

¹⁰In fact AD contradicts $\text{AC}_{\omega_1}(\mathbb{R})$, since this choice principle implies the failure of PSP.

is injective, and ${}^{<\lambda}X \mapsto \lambda \times {}^{<\lambda}X \mapsto {}^{<\lambda}X \times {}^{<\lambda}X \mapsto {}^{<\lambda}({}^{<\lambda}X)$, so we are done by the Schröder-Bernstein theorem. \square

2.4. Algebras of sets. Let X be an arbitrary set and let $\alpha \geq \omega$. A collection $\mathcal{A} \subseteq \mathcal{P}(X)$ is called **α -algebra (on X)** if it is closed under complements and well-ordered unions of length $< \alpha$, that is $\bigcup_{\nu < \beta} A_\nu \in \mathcal{A}$ for every $\beta < \alpha$ and every sequence $\langle A_\nu \mid \nu < \beta \rangle$ of sets in \mathcal{A} . An ω -algebra is usually called an algebra, and an $\alpha + 1$ -algebra is the same as an α^+ -algebra. In particular, an $\omega + 1$ -algebra (i.e. an ω_1 -algebra) is what is usually called a σ -algebra.

Given a family $\mathcal{G} \subseteq \mathcal{P}(X)$, the **α -algebra (on X) generated by \mathcal{G}** is the smallest α -algebra \mathcal{A} on X such that $\mathcal{G} \subseteq \mathcal{A}$, and is denoted by

$$\text{Alg}(\mathcal{G}, \alpha).$$

Thus the family of the Borel subsets of a topological space (X, τ) is

$$\mathbf{B}(X) = \text{Alg}(\tau, \omega + 1) = \text{Alg}(\tau, \omega_1).$$

The algebra $\text{Alg}(\mathcal{G}, \alpha)$ can be inductively generated as follows: let

$$\Sigma_1(\mathcal{G}, \alpha) := \mathcal{G}$$

$$\Pi_1(\mathcal{G}, \alpha) := \{X \setminus A \mid A \in \mathcal{G}\}$$

and for $\gamma > 1$

$$\Sigma_\gamma(\mathcal{G}, \alpha) := \left\{ \bigcup_{\nu < \beta} A_\nu \mid \beta < \alpha \wedge \forall \nu < \beta (A_\nu \in \bigcup_{\xi < \gamma} \Pi_\xi(\mathcal{G}, \alpha)) \right\}$$

$$\Pi_\gamma(\mathcal{G}, \alpha) := \{X \setminus A \mid A \in \Sigma_\gamma(\mathcal{G}, \alpha)\}.$$

The next result is straightforward.

LEMMA 2.2. *If $1 \leq \gamma < \delta$ and $\delta \geq 3$ we have that*

$$\Sigma_\gamma(\mathcal{G}, \alpha) \cup \Pi_\gamma(\mathcal{G}, \alpha) \subseteq \Sigma_\delta(\mathcal{G}, \alpha) \cap \Pi_\delta(\mathcal{G}, \alpha)$$

so that if $\mathcal{G} \subseteq \Sigma_2(\mathcal{G}, \alpha)$ the inclusion holds for every $1 \leq \gamma < \delta$. Moreover

$$\text{Alg}(\mathcal{G}, \alpha) = \bigcup_{\gamma \in \text{Ord}} \Sigma_{1+\gamma}(\mathcal{G}, \alpha) = \bigcup_{\gamma \in \text{Ord}} \Pi_{1+\gamma}(\mathcal{G}, \alpha).$$

Notice that Ord in the equation above can be replaced by any cardinal λ with $\text{cof}(\lambda) \geq \alpha$.

LEMMA 2.3. *Suppose X is an arbitrary set, α is an infinite ordinal, $\lambda \geq \alpha$ is a regular cardinal, and $\mathcal{G} \subseteq \mathcal{P}(X)$ contains at least two elements. Then*

(a) ${}^{<\lambda}\mathcal{G} \twoheadrightarrow \text{Alg}(\mathcal{G}, \alpha)$, and

(b) if $\lambda = \nu^+$ then ${}^\nu\mathcal{G} \twoheadrightarrow \text{Alg}(\mathcal{G}, \alpha)$.

In particular, if AC holds and $|\mathcal{G}| \leq \mu$ then $|\text{Alg}(\mathcal{G}, \lambda)| \leq \mu^{<\lambda}$, which equals to $\mu^{|\nu|}$ if (b) holds.

PROOF. (a) It is enough to construct surjections

$$p_\gamma: {}^{<\lambda}\mathcal{G} \twoheadrightarrow \Sigma_{1+\gamma}(\mathcal{G}, \alpha)$$

for each $\gamma < \lambda$, so that

$$\lambda \times {}^{<\lambda}\mathcal{G} \rightarrow \bigcup_{\gamma < \lambda} \Sigma_{1+\gamma}(\mathcal{G}, \alpha), \quad (\gamma, s) \mapsto p_\gamma(s)$$

is a surjection, and hence the result follows from Proposition 2.1. The construction of $p_0: {}^{<\lambda}\mathcal{G} \rightarrow \Sigma_1(\mathcal{G}, \alpha) = \mathcal{G}$ is immediate. Suppose p_ν has been defined for all $\nu < \gamma$, and let

$$q_\gamma: \gamma \times {}^{<\lambda}\mathcal{G} \twoheadrightarrow \bigcup_{\nu < \gamma} \Sigma_{1+\nu}(\mathcal{G}, \alpha), \quad (\nu, s) \mapsto p_\nu(s).$$

Since $\gamma \times {}^{<\lambda}\mathcal{G} \subseteq \lambda \times {}^{<\lambda}\mathcal{G}$, by repeated applications of Proposition 2.1 one gets ${}^{<\lambda}\mathcal{G} \twoheadrightarrow \bigcup_{\nu < \gamma} \Sigma_{1+\nu}(\mathcal{G}, \alpha)$, and hence ${}^{<\lambda}({}^{<\lambda}\mathcal{G}) \twoheadrightarrow \Sigma_{1+\gamma}(\mathcal{G}, \alpha)$, and therefore ${}^{<\lambda}\mathcal{G} \twoheadrightarrow \Sigma_{1+\gamma}(\mathcal{G}, \alpha)$.

(b) As in part (a) one constructs surjections $p_\gamma: {}^\nu\mathcal{G} \twoheadrightarrow \Sigma_{1+\gamma}(\mathcal{G}, \alpha)$ for all $\gamma < \nu^+$, using the fact that ${}^\nu 2 \twoheadrightarrow \nu^+$. \square

The σ -algebra of the **Borel subsets** of a topological space (X, τ) is $\mathbf{B}(X, \tau) = \text{Alg}(\tau, \omega + 1)$. When the topology τ is clear from the context, it is customary write $\mathbf{B}(X)$, $\Sigma_\alpha^0(X)$ and $\Pi_\alpha^0(X)$ instead of $\mathbf{B}(X, \tau)$, $\Sigma_\alpha(\tau, \omega + 1)$ and $\Pi_\alpha(\tau, \omega + 1)$. Thus $\Sigma_1^0(X)$ is the family τ of all open sets of X , $\Pi_1^0(X)$ is the family of all closed subsets of X , $\Sigma_2^0(X)$ is the collection of all \mathbf{F}_σ subsets of X , Π_2^0 is the collection of all \mathbf{G}_δ subsets of X , and so on. Assuming ω_1 is regular¹¹ (a fact that follows from $\text{AC}_\omega(\mathbb{R})$), then $\mathbf{B}(X) = \bigcup_{1 \leq \alpha < \omega_1} \Sigma_\alpha^0(X) = \bigcup_{1 \leq \alpha < \omega_1} \Pi_\alpha^0(X)$. If X is metrizable then every closed set is \mathbf{G}_δ , so $\Sigma_\alpha^0(X) \cup \Pi_\alpha^0(X) \subseteq \Sigma_\beta^0(X) \cap \Pi_\beta^0(X)$ for every $1 \leq \alpha < \beta$. From Lemma 2.3 we obtain at once the following result.

COROLLARY 2.4. *Assume $\text{AC}_\omega(\mathbb{R})$. If τ is a topology on $X \neq \emptyset$, then ${}^\omega\tau \twoheadrightarrow \mathbf{B}(X)$.*

2.5. Descriptive set theory.

2.5.1. Polish spaces. A topological space is **perfect** if it has no isolated points; it is **zero-dimensional** if it has a basis of clopen sets. A **Polish space** is a separable, completely metrizable topological space. If the spaces X_n ($n \in \omega$) are Polish, then so is $\prod_n X_n$ with the product topology. Since any countable set with the discrete topology is Polish, then the **Cantor space** ${}^\omega 2$ and the **Baire space** ${}^\omega\omega$ are Polish: a countable dense set is given by the sequences that are eventually constant, and a complete metric for them is given by $d(x, y) = 2^{-n}$ if n is least such that $x(n) \neq y(n)$ and $d(x, y) = 0$ if there is no such n . This metric is in fact an ultrametric, so the balls are clopen and hence these spaces are zero-dimensional. Recall that a metric d on a space X is called **ultrametric** if it satisfies the following strengthening of the triangular inequality: $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$. The space ${}^\omega 2$ is the unique (up to homeomorphism) nonempty compact zero-dimensional perfect Polish space, and ${}^\omega\omega$ is the unique (up to homeomorphism) nonempty zero-dimensional perfect Polish space in which all compact subsets have empty interior [Kec95, Theorems 7.4 and 7.7]. Any two uncountable Polish spaces are Borel isomorphic [Kec95, Theorem 15.6] and since many questions in descriptive set theory are invariant under Borel isomorphism, it is customary to use \mathbb{R} to denote any uncountable Polish space.

2.5.2. Pointclasses. A **general pointclass** Γ is an operation (i.e. a class-functions) assigning to every nonempty topological space X a nonempty family $\Gamma(X) \subseteq \mathcal{P}(X)$. The **dual** of Γ is the general pointclass defined by $\check{\Gamma}(X) := \{X \setminus A \mid A \in \Gamma(X)\}$. The **ambiguous general pointclass** associated to Γ (or to $\check{\Gamma}$) is the general pointclass Δ_Γ defined by $\Delta_\Gamma(X) := \Gamma(X) \cap \check{\Gamma}(X)$. A general pointclass Γ is **hereditary** if $\Gamma(Y) = \{A \cap Y \mid A \in \Gamma(X)\}$, for every pair of nonempty topological spaces $Y \subseteq X$.

A general pointclass Γ is said to be **boldface** if it is closed under continuous preimages, that is to say: if $f: X \rightarrow Y$ is continuous and $B \in \Gamma(Y)$ then $f^{-1}(B) \in \Gamma(X)$. When the topological space is clear from the context, we write $A \in \Gamma$ rather than $A \in \Gamma(X)$. General boldface pointclasses are usually denoted by Greek letters such as $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$, variously decorated, and are typeset boldface, whence the name. Examples of general boldface pointclasses are: the collection of all open sets Σ_1^0 , its dual pointclass Π_1^0 i.e. the collection of all closed sets, its ambiguous pointclass Δ_1^0 i.e. the collection of all clopen sets, the Borel pointclasses Σ_α^0 , Π_α^0 and the Borel sets \mathbf{B} .

¹¹It is consistent with ZF that \mathbb{R} is countable union of countable sets, and hence that every subset of \mathbb{R} is Borel.

A **(boldface) pointclass** is a general (boldface) pointclass restricted to *Polish* spaces. In other words, a pointclass is an operation assigning to every nonempty Polish space X a nonempty family $\Gamma(X) \subseteq \mathcal{P}(X)$, and it is boldface if $f^{-1}(B) \in \Gamma(X)$ for every $B \in \Gamma(Y)$ and every continuous function $f: X \rightarrow Y$ between Polish spaces. If $\Gamma(\omega\omega) \neq \mathcal{P}(\omega\omega)$ (equivalently: $\Gamma(X) \neq \mathcal{P}(X)$ for any *uncountable* Polish space X), then Γ is called a **proper** pointclass. If $\{\emptyset, X\} \subset \Gamma(X)$ for some space X (equivalently: $\Gamma(\omega\omega) \supseteq \Delta_1^0(\omega\omega)$) then Γ is **nontrivial**. The boldface pointclass Γ is called **nonselfdual** if $\Gamma(\omega\omega) \neq \check{\Gamma}(\omega\omega)$ (equivalently: $\Gamma(X) \neq \check{\Gamma}(X)$ for any *uncountable* Polish space X), and **selfdual** otherwise. The boldface pointclass Γ admits a **universal set** if there is $\mathcal{U} \in \Gamma(\omega\omega \times \omega\omega)$ such that $\Gamma(\omega\omega) = \{\mathcal{U}^{(y)} \mid y \in \omega\omega\}$, where $\mathcal{U}^{(y)} := \{x \in \omega\omega \mid (x, y) \in \mathcal{U}\}$ is the **horizontal section** of \mathcal{U} . If Γ admits a universal set, then it is nonselfdual: under AD these two properties become equivalent, that is: Γ admits a universal set if and only if it is nonselfdual. These definitions extend to the case of a *general* boldface pointclass: Γ is proper (respectively, nontrivial, nonselfdual, selfdual) if its restriction to the class of Polish spaces is a proper (respectively, nontrivial, nonselfdual, selfdual) boldface pointclass.

Besides the Borel sets \mathbf{B} , and the Borel pointclasses $\Sigma_\alpha^0, \Pi_\alpha^0$, examples of boldface pointclasses are the projective pointclasses Σ_n^1 (for $n \in \omega$), and the pointclasses associated to third-order arithmetic, such as Σ_1^2 . As usual, the dual of Σ_α^i is Π_α^i and its ambiguous part is Δ_α^i (for $i = 0, 1, 2$ and $0 \neq \alpha < \omega_1$). The pointclasses Σ_α^i and Π_α^i are hereditary and nonselfdual, the pointclass \mathbf{B} is hereditary and selfdual, while the Δ_α^i 's are selfdual and not hereditary. The projective pointclasses are *not* general boldface pointclasses, as they are not closed under continuous preimages. To see this consider the pointclass Σ_1^1 of analytic sets, i.e. $\Sigma_1^1(X) = \{f(\omega\omega) \mid f: \omega\omega \rightarrow X \text{ is continuous}\}$. If X is a Polish space and Y is a proper Π_1^1 subset of X , then Y is the preimage of $X \in \Sigma_1^1(X)$ under the inclusion map $Y \hookrightarrow X$, but Y is not a continuous image of the Baire space, ie. $Y \notin \Sigma_1^1(Y)$.

2.5.3. The prewellordering and scale properties. Let A be a set. A **(regular) norm** on A is a map $\rho: A \rightarrow \text{Ord}$ which is surjective onto some $\alpha \in \text{Ord}$. The ordinal α is called **length** of ρ . To each norm $\rho: A \rightarrow \alpha$ we can canonically associate a **prewellordering** (i.e. a reflexive, transitive, connected, and well-founded relation) \preceq_ρ of A by setting $x \preceq_\rho y \Leftrightarrow \rho(x) \leq \rho(y)$ for all $x, y \in A$. Conversely, to every prewellordering \preceq of A we can canonically associate a (unique) regular norm $\rho: A \rightarrow \alpha$ (for some $\alpha \in \text{Ord}$) such that $\preceq = \preceq_\rho$: in this case we say that \preceq has length α .

Given a boldface pointclass Γ and a Polish space X , a norm ρ on $A \in \Gamma(X)$ is a **Γ -norm** if there are two binary relations $\leq_\rho^\Gamma \in \Gamma(X \times X)$ and $\leq_\rho^{\check{\Gamma}} \in \check{\Gamma}(X \times X)$ on X such that for all $y \in A$

$$\forall x \in X \left[(x \in A \wedge \rho(x) \leq \rho(y)) \Leftrightarrow x \leq_\rho^\Gamma y \Leftrightarrow x \leq_\rho^{\check{\Gamma}} y \right].$$

In other words: the initial segments of the prewellordering \preceq_ρ are uniformly Δ_Γ . We say that the pointclass Γ is **normed** or has the **prewellordering property** if every $A \in \Gamma(X)$ admits a Γ -norm, for all Polish spaces X . A Γ -norm on a set $A \in \Delta_\Gamma$ is automatically a $\check{\Gamma}$ -norm as well. On the other hand the property of being normed does not pass to the dual: for example Π_1^1 is normed, but Σ_1^1 is not [Mos09, Section 4.B].

Let X be a topological space. A **scale** on $A \subseteq X$ is a sequence $\langle \rho_n \mid n \in \omega \rangle$ of (regular) norms on A such that for every sequence $\langle x_n \mid n \in \omega \rangle$ of points from A , if the x_n 's converge to $x \in X$ and for every $n \in \omega$ the sequence $\langle \rho_n(x_i) \mid i \in \omega \rangle$ is eventually equal to some $\lambda_n \in \text{Ord}$, then $x \in A$ and $\rho_n(x) \leq \lambda_n$ for all $n \in \omega$. When $X = \omega\omega$, the existence of a scale $\langle \rho_n \mid n \in \omega \rangle$ on A is equivalent to the assertion that A is the projection of a tree on $\omega \times \kappa$, where $\kappa := \sup_n \text{ran}(\rho_n)$ (see Section 2.6.1 for the relevant definitions).

Given a boldface pointclass $\mathbf{\Gamma}$ closed under countable intersections and countable unions, a scale $\langle \rho_n \mid n \in \omega \rangle$ on a set $A \in \mathbf{\Gamma}(X)$ is called **$\mathbf{\Gamma}$ -scale** if all the ρ_n 's are $\mathbf{\Gamma}$ -norms.¹² The pointclass $\mathbf{\Gamma}$ has the **scale property** if every $A \in \mathbf{\Gamma}(X)$ admits a $\mathbf{\Gamma}$ -scale for all Polish spaces X .

2.6. Trees and reductions.

2.6.1. *Graphs and trees.* A **graph** $G = (V, E)$ consists of a nonempty set V of vertices and a set $E \subseteq [V]^2$ of edges. Whenever V is clear from the context, the graph is identified with E , and we write $v_0 G v_1$ or $v_0 E v_1$ instead of $\{v_0, v_1\} \in E$. A **combinatorial tree** is a connected acyclic graph. If a specific vertex is singled out, the resulting object is a **rooted combinatorial tree** and the chosen vertex is called **root**. The size of a combinatorial tree is the cardinality of the set V of its vertices. As the nature of the elements of V is irrelevant, a combinatorial tree of size κ can be construed as a set of edges $E \subseteq [\kappa]^2$, and hence it can be identified with an element of ${}^{\kappa \times \kappa} 2$ (via its characteristic function). Thus the space of all combinatorial trees of size κ is

$$(2.5) \quad \text{CT}_\kappa := \{f \in {}^{\kappa \times \kappa} 2 \mid f \text{ satisfies (2.6a)–(2.6c)}\}$$

with

$$(2.6a) \quad \forall \alpha, \beta \in \kappa \ (f(\alpha, \alpha) = 0 \wedge (f(\alpha, \beta) = 1 \Rightarrow f(\beta, \alpha) = 1))$$

$$(2.6b) \quad \forall \alpha, \beta \in \kappa \ [f(\alpha, \beta) = 1 \vee \exists s \in {}^{<\omega} \kappa \setminus \{\emptyset\} \ \forall i < \text{lh}(s) - 1 \ (f(s(i), s(i+1)) = 1 \wedge f(\alpha, s(0)) = 1 \wedge f(s(\text{lh}(s) - 1), \beta) = 1)]$$

$$(2.6c) \quad \forall s \in {}^{<\omega} \kappa \ (\text{lh}(s) \geq 3 \wedge \forall i < \text{lh}(s) - 1 \ (f(s(i), s(i+1)) = 1) \Rightarrow f(s(\text{lh}(s) - 1), s(0)) = 0).$$

The intuition behind these formulæ is that: (2.6a) says that f codes a graph G_f , (2.6b) says that G_f is connected, while (2.6c) says that G_f is acyclic. It is easy to check that CT_ω is a \mathbf{G}_δ subset of ${}^{\omega \times \omega} 2$ (which is homeomorphic to the Cantor space ${}^\omega 2$), and hence CT_ω is a Polish space.

2.6.2. *Descriptive set-theoretic trees.* The word *tree* may also refer to a different concept in descriptive set theory: a **tree on a set** X is a nonempty subset of ${}^{<\omega} X$ closed under initial segments (ordered by the prefix relation). Such an object is often called a **descriptive set-theoretic tree**, and

$$(2.7) \quad \text{Tr}(X)$$

is the set of all descriptive set-theoretic trees on X . Any descriptive set-theoretic tree can be seen as a rooted combinatorial tree with root \emptyset , and conversely. Elements of a (descriptive set-theoretic) tree T on X are called **nodes**. We say that T is **pruned** if every node has a proper extension, and that T is **$< \kappa$ -branching** (for κ a cardinal) if every node has $< \kappa$ -many immediate successors, that is $|\{x \in X \mid s \frown x \in T\}| < \kappa$ for every $s \in T$. Sometimes, $< (\kappa + 1)$ -branching trees are simply called κ -branching. The **body** of T is the set of all infinite branches of T , that is the set

$$[T] := \{f \in {}^\omega X \mid \forall n \ (f \upharpoonright n \in T)\}.$$

If T is a tree on $X \times Y$ then the nodes are construed as pairs of sequences $\langle \langle x_0, \dots, x_n \rangle, \langle y_0, \dots, y_n \rangle \rangle$ rather than sequences of pairs $\langle \langle x_0, y_0 \rangle, \dots, \langle x_n, y_n \rangle \rangle$, and similarly the elements of $[T]$ are construed as pairs $(f, g) \in {}^\omega X \times {}^\omega Y$ such that $(f \upharpoonright n, g \upharpoonright n) \in T$ for all n . The **projection** (on the first coordinate) of a subset $A \subseteq X \times Y$ of a cartesian product is

$$(2.8) \quad \text{p} A = \{x \in X \mid \exists y \in Y \ (x, y) \in A\}$$

¹²The definition of $\mathbf{\Gamma}$ -scale for general $\mathbf{\Gamma}$'s requires that the norms be *uniformly* in $\mathbf{\Gamma}$ — see [Mos09, p. 173].

The projection of a tree T on $X \times Y$ is the set

$$\text{p}[T] := \{f \in {}^\omega X \mid \exists g \in {}^\omega Y (f, g) \in [T]\}.$$

2.6.3. Quasi-orders and equivalence relations. A binary relation R on a set X is called **quasi-order** or **preorder** if it is reflexive and transitive, and a symmetric quasi-order is called **equivalence relation**. The set X is called **domain of R** and it is denoted by $\text{dom}(R)$. If $Y \subseteq X = \text{dom}(R)$, we denote by $R \upharpoonright Y$ the restriction of R to Y , i.e. the quasi-order $R \cap Y^2$. If R is a quasi-order, then $R^{-1} := \{(y, x) \mid (x, y) \in R\}$ is also a quasi-order, and $E_R := R \cap R^{-1}$ is the equivalence relation induced by R . A quasi-order R canonically induces a partial order on the quotient space X/E_R , which is called **quotient order of R** .

If E is an equivalence relation on X , then $[x]_E = [x]$ is the **equivalence class of $x \in X$** and X/E is the **quotient space**. A set $A \subseteq X = \text{dom}(E)$ is **invariant under E** if $y \in A$ whenever $x E y$ for some $x \in A$. The **E -saturation of $A \subseteq X$** , in symbols $[A]_E$ or even just $[A]$, is the smallest invariant set containing A , that is

$$[A]_E = [A] := \bigcup_{x \in A} [x].$$

Given a (general) boldface pointclass Γ , we say that a quasi-order (or, more generally: a binary relation) R on X is in Γ if $R \in \Gamma(X \times X)$. Notice that if Γ is closed under finite intersections, this implies that the associated equivalence relation E_R is in Γ as well.

2.6.4. Reducibility. Given two quasi-orders R, S on the sets X, Y , we say that R **reduces** to S (in symbols $R \leq S$) if and only if there is a function $f: X \rightarrow Y$ which reduces R to S , i.e. such that for all $x, x' \in X$

$$x R x' \Leftrightarrow f(x) S f(x').$$

Notice that reducibility is just (full) homomorphism, the name is just for historical reasons and because we usually impose some restriction on the functions that can be used as reductions. If $R \leq S$ and $S \leq R$ we say that R and S are **bi-reducible** (in symbols $R \sim S$). Notice that if E is an equivalence relation and $R \leq E$, then R is an equivalence relation as well. If \mathcal{C} is a class of quasi-orders, we say that the quasi-order R is **complete for \mathcal{C}** if $S \leq R$ for every $S \in \mathcal{C}$. By limiting f to range in a given collection of functions we obtain a restricted form of reducibility. For example if f is Borel or $f \in L(\mathbb{R})$, then the notions of **Borel reducibility \leq_B** and **$L(\mathbb{R})$ -reducibility $\leq_{L(\mathbb{R})}$** are obtained, respectively. If \leq_* is a restricted form of reducibility, we say that R is **\leq_* -complete for \mathcal{C}** if $S \leq_* R$ for every $S \in \mathcal{C}$. In particular R is **Borel-complete** for \mathcal{C} if for every $S \in \mathcal{C}$ there is a Borel function that witnesses $S \leq R$, i.e. $S \leq_B R$.

REMARK 2.5. Our definition of “ R is (\leq_*) -complete for \mathcal{C} ” does not require that $R \in \mathcal{C}$. In fact, in most applications we will have that \mathcal{C} is a collection of quasi-orders on some Polish space, while R will be the embeddability relation on some elementary class of *uncountable* models.

The notion of reducibility between equivalence relations and quasi-orders is intimately related to the notion of cardinality of a set. If X, Y are arbitrary sets and $\text{id}(X)$ and $\text{id}(Y)$ denote, respectively, the equality relations on X and Y , then

$$|X| \leq |Y| \Leftrightarrow \text{id}(X) \leq \text{id}(Y)$$

and hence

$$|X| = |Y| \Leftrightarrow \text{id}(X) \sim \text{id}(Y).$$

Thus reducibility between equivalence relations can be seen as a generalization of the notion of cardinality. Moreover, if E and F are equivalence relations on X and Y , respectively, then

$$E \leq F \Rightarrow |X/E| \leq |Y/F|$$

and

$$E \sim F \Rightarrow |X/E| = |Y/F|.$$

The converse implications hold if we can find inverses for the surjections $X \twoheadrightarrow X/E$. This is a trivial matter under AC, while if choice is not available and $X = Y = \mathbb{R}$, we may appeal to uniformization. For example the implications above can be reversed if we assume $\text{AD} + \text{V} = \text{L}(\mathbb{R})$ and E, F are Δ_1^2 , or if we assume $\text{AD}_{\mathbb{R}}$ and E and F are arbitrary equivalence relations on \mathbb{R} .

If a set A is the surjective image of X via some map f then, letting E_A be the equivalence relation on X defined by $x E_A x' \Leftrightarrow f(x) = f(x')$, the factoring map $f: X/E_A \rightarrow A$, $[x]_{E_A} \mapsto f(x)$ is well-defined and witnesses $|X/E_A| = |A|$. This observation can be turned into a method to compute cardinalities. For example, in models of AD one is mainly interested in the cardinalities of sets which are surjective images of \mathbb{R} , and such cardinalities are called **small cardinalities**, or **small cardinals** when we are dealing with well-orderable sets. If A, B are two sets whose cardinality is small, then $E_A \leq E_B \Rightarrow |A| \leq |B|$ (again, in some special situations this implication can be reversed).

3. The generalized Cantor space

3.1. Basic facts. The usual topology on the Cantor space ${}^\omega 2$ is the product topology, which is generated by the sets

$$N_s^\omega := \{x \in {}^\omega 2 \mid s \subseteq x\}$$

for $s \in {}^{<\omega} 2$. If ω is replaced by some uncountable cardinal κ , then there are at least two topologies on the **generalized Cantor space** ${}^\kappa 2$ that can claim to be the natural generalization of the previous construction.

DEFINITION 3.1. Let κ be an infinite cardinal, and let

$$N_s^\kappa := \{x \in {}^\kappa 2 \mid s \subseteq x\},$$

where $s: u \rightarrow 2$ and $u \subseteq \kappa$.

- The **bounded topology** τ_b on ${}^\kappa 2$ is the one generated by the collection

$$\mathcal{B}_b := \{N_s^\kappa \mid s \in {}^{<\kappa} 2\}.$$

- The **product topology** τ_p on ${}^\kappa 2$ is the product of κ copies of 2 with the discrete topology, and hence it is generated by the collection

$$\mathcal{B}_p := \{N_s^\kappa \mid s \in \mathbf{Fn}(\kappa, 2; \omega)\}.$$

The families \mathcal{B}_b and \mathcal{B}_p are called the **canonical basis** for τ_b and τ_p , respectively, and their elements are called the **basic open sets**.

To simplify the notation, when κ is clear from the context we write N_s instead of N_s^κ . Conversely, we write $\tau_b({}^\kappa 2)$, $\mathcal{B}_b({}^\kappa 2)$, $\tau_p({}^\kappa 2)$, and $\mathcal{B}_p({}^\kappa 2)$ instead of τ_b , \mathcal{B}_b , τ_p , and \mathcal{B}_p if attention must be paid to the underlying space. When $\kappa = \omega$ the topologies τ_b and τ_p coincide, so when dealing with the Cantor space ${}^\omega 2$ we can talk about topology without further comments. In contrast, Lemma 3.10 below shows that τ_b and τ_p are significantly different when κ is uncountable. For example

$$(3.1) \quad \kappa \text{ regular} \Rightarrow \tau_b \text{ is closed under intersections of size } < \kappa$$

while τ_p is never closed under countable intersections — see Proposition 3.12(i) below.

Generalizing the notion seen in Section 2.6.2, a **descriptive set-theoretic tree on X of height κ** is a nonempty $T \subseteq {}^{<\kappa} X$ closed under initial segments. Such T is called **pruned** if

$$\forall t \in T \forall \alpha < \kappa \exists t' \in T [\text{lh}(t') = \alpha \wedge (t \subseteq t' \vee t' \subseteq t)].$$

Note that $\emptyset \neq C \subseteq {}^\kappa 2$ is closed with respect to τ_b if and only if it is of the form

$$[T] := \{x \in {}^\kappa 2 \mid \forall \alpha < \kappa (x \restriction \alpha \in T)\}$$

with T a pruned tree on 2 of height κ .

The literature on ${}^\kappa 2$ with κ an uncountable cardinal (and on some other strictly related spaces, see Sections 6.1 and 7.2) falls into two camps: most of the papers in general topology (see e.g. [Sto48, Kur66, Ünl82, TT93, CM97, CGP98, Kra01, Ili12, CK13]) deal with τ_p or with the so-called box topology, while papers on infinitary logics (see [Vau75, MV93, Hal96, SV00, HS01, She01, SV02, She04, DV11, FHK14]) use almost exclusively the bounded topology. In this paper, instead, both the bounded and the product topology play an important role: although the statements of our main results refer to the bounded topology, several proofs make an essential use of (an homeomorphic copy of) the space $({}^\kappa 2, \tau_p)$ — see Sections 12 and 13. This seems to be a curious feature, and we currently do not know if the use of the product topology can be avoided at all.

REMARKS 3.2. (i) The name bounded topology comes from the fact that τ_b can be equivalently defined as the topology generated by the collection of all sets of the form N_s^κ with $s: u \rightarrow 2$ for some *bounded* $u \subseteq \kappa$: such a collection is an alternative basis for τ_b which properly contains \mathcal{B}_b . When κ is regular, this basis can be also described as the collection of all sets of the form N_s^κ with $s: u \rightarrow 2$ for some $u \subset \kappa$ of *size* $< \kappa$ (thus avoiding any reference to the ordering of κ in the definition of τ_b): however, this is not true when κ is singular because if $u \subseteq \kappa$ is *cofinal* in κ , then every set N_s^κ for $s: u \rightarrow 2$ is a proper τ_b -closed set.

(ii) The sets N_s^κ with $s: \{\alpha\} \rightarrow 2$ for $\alpha \in \kappa$, form a subbasis $\tilde{\mathcal{B}}_p$ (which generates \mathcal{B}_p) for the product topology on ${}^\kappa 2$. For ease of notation the elements of $\tilde{\mathcal{B}}_p$ are denoted by

$$\tilde{N}_{\alpha,i}^\kappa = \tilde{N}_{\alpha,i} := \{x \in {}^\kappa 2 \mid x(\alpha) = i\}.$$

(iii) The definition of product topology makes sense for spaces of the form A^2 (or even A^X for X a topological space) with A an arbitrary set, while the definition of the bounded topology requires that A be well-orderable — see Section 7.2.

By Remark 3.2(i), if κ is regular then the bounded topology is generated by the sets N_s^κ with $s \in \mathbf{Fn}(\kappa, 2; \kappa)$. This suggests that when κ is regular τ_p and τ_b lie at the extrema of a spectrum of topologies on ${}^\kappa 2$.

DEFINITION 3.3. Let $\lambda \leq \kappa$ be infinite cardinals. The λ -**topology** τ_λ on ${}^\kappa 2$ is generated by the basis

$$\mathcal{B}_\lambda := \{N_s^\kappa \mid s \in \mathbf{Fn}(\kappa, 2; \lambda)\}.$$

In particular: $\tau_p = \tau_\omega$.

As usual, when we need to explicitly refer to the underlying space we write $\tau_\lambda({}^\kappa 2)$ and $\mathcal{B}_\lambda({}^\kappa 2)$ instead of, respectively, τ_λ and \mathcal{B}_λ . It is easy to check that the sets in $\mathcal{B}_p, \mathcal{B}_\lambda, \mathcal{B}_b$ are clopen, and hence the topologies $\tau_p, \tau_\lambda, \tau_b$ are zero-dimensional.

LEMMA 3.4. *Let κ be an uncountable cardinal, and consider the topologies $\tau_p, \tau_\lambda, \tau_b$ on ${}^\kappa 2$.*

- (a) *if $\omega \leq \lambda < \nu < \kappa$ are cardinals, then $\mathcal{B}_\lambda \subset \mathcal{B}_\nu$, each $N_s^\kappa \in \mathcal{B}_\nu$ is τ_λ -closed, and $\tau_\lambda \subset \tau_\nu$,*
- (b) *$\tau_{\text{cof}(\kappa)} \subseteq \tau_b$, and*
- (c) *$\tau_b = \tau_{\text{cof}(\kappa)} \Leftrightarrow \kappa = \text{cof}(\kappa)$.*

PROOF. (a) It is clear that $\mathcal{B}_\lambda \subset \mathcal{B}_\nu$ and hence $\tau_\lambda \subseteq \tau_\nu$. Fix $N_s \in \mathcal{B}_\nu$. If $|\text{dom } s| < \lambda$, then N_s is τ_λ -clopen, if $\lambda \leq |\text{dom } s| < \nu$, then $N_s = \bigcap_{\alpha \in \text{dom } s} \tilde{N}_{\alpha, s(\alpha)}^\kappa$ is τ_p -closed and hence τ_λ -closed, but it is not τ_λ -open as it does not contain any element of \mathcal{B}_λ . Therefore $\tau_\lambda \neq \tau_\nu$.

(b) To see that $\tau_{\text{cof}(\kappa)} \subseteq \tau_b$, it is enough to show that $\mathcal{B}_{\text{cof}(\kappa)} \subseteq \tau_b$: if $s \in \mathbf{Fn}(\kappa, 2; \text{cof}(\kappa))$ then $N_s = \bigcup \{N_t \mid t \supseteq s \wedge t \in {}^\alpha 2\}$, where $\alpha = \sup \text{dom}(s) < \kappa$.

(c) If κ is singular, and $s \in {}^\alpha 2$ with $\text{cof}(\kappa) \leq \alpha < \kappa$, then $N_s \in \mathcal{B}_b$, but N_s does not contain any set in $\mathcal{B}_{\text{cof}(\kappa)}$, so it is not $\tau_{\text{cof}(\kappa)}$ -open. If κ is regular, then $\mathcal{B}_b \subset \mathcal{B}_\kappa$ so $\tau_b \subseteq \tau_\kappa$, and therefore $\tau_b = \tau_\kappa$. \square

In Sections 6 and 8 we use these topologies that lie strictly between the product topology and the bounded topology, that is τ_λ with $\omega < \lambda < \min(\text{cof}(\kappa)^+, \kappa)$, while we have no use for τ_λ with $\text{cof}(\kappa) < \lambda < \kappa$. Most of the nontrivial results on these intermediate topologies seem to require the axiom of choice.

In order to simplify the notation, when κ is regular we stipulate the following

CONVENTION 3.5. When κ is regular, the canonical basis for τ_κ is taken to be \mathcal{B}_b , that is

$$\mathcal{B}_\kappa := \{N_s^\kappa \mid s \in {}^{<\kappa} 2\}.$$

3.2. * More on 2^κ .

3.2.1. Lipschitz and continuous reductions.

DEFINITION 3.6. Let κ be an infinite cardinal. A function $\varphi: {}^{<\kappa} 2 \rightarrow {}^{<\kappa} 2$ is

- **monotone** if $\forall s, t \in {}^{<\kappa} 2 (s \subseteq t \Rightarrow \varphi(s) \subseteq \varphi(t))$,
- **Lipschitz** if it is monotone and $\forall s \in {}^{<\kappa} 2 (\text{lh}(\varphi(s)) = \text{lh}(s))$,
- **continuous** if it is monotone and $\text{lh}(\bigcup_{\alpha < \kappa} \varphi(x \upharpoonright \alpha)) = \kappa$, for all $x \in {}^\kappa 2$.

If φ is Lipschitz then it is continuous, and if φ is Lipschitz or continuous, then

$$f_\varphi: {}^\kappa 2 \rightarrow {}^\kappa 2, \quad x \mapsto \bigcup_{\alpha < \kappa} \varphi(x \upharpoonright \alpha)$$

is the **function induced by φ** .

LEMMA 3.7. A function $f: {}^\kappa 2 \rightarrow {}^\kappa 2$ is continuous with respect to τ_b if and only if it is of the form f_φ for some continuous $\varphi: {}^{<\kappa} 2 \rightarrow {}^{<\kappa} 2$.

PROOF. If f is τ_b -continuous, then the map φ defined by setting for $s \in {}^{<\kappa} 2$

$$\varphi(s) := \text{the longest } t \text{ with length } \leq \text{lh}(s) \text{ such that } f(N_s) \subseteq N_t$$

is continuous and $f = f_\varphi$. Conversely if $f = f_\varphi$ with φ continuous then $f^{-1}(N_t) = \bigcup_{\varphi(s) \supseteq t} N_s$, and hence f is τ_b -continuous. \square

We call a function $f: {}^\kappa 2 \rightarrow {}^\kappa 2$ **Lipschitz** if it is of the form f_φ for some Lipschitz φ or, equivalently, if it is such that for all $x, y \in {}^\kappa 2$ and $\alpha < \kappa$, $x \upharpoonright \alpha = y \upharpoonright \alpha \Rightarrow f(x) \upharpoonright \alpha = f(y) \upharpoonright \alpha$. (The reason for this terminology is that when $\kappa = \omega$ then f is Lipschitz if and only if $d(f(x), f(y)) \leq d(x, y)$, where d is the usual metric on ${}^\omega 2$.) It is immediate to check that the composition of Lipschitz functions is Lipschitz, and that Lipschitz functions are τ_b -continuous.

DEFINITION 3.8. Let $A, B \subseteq {}^\kappa 2$. We say that A is **Lipschitz reducible** to B , in symbols

$$A \leq_L^\kappa B,$$

if $A = f^{-1}(B)$ for some Lipschitz $f: {}^\kappa 2 \rightarrow {}^\kappa 2$. Equivalently, $A \leq_L^\kappa B$ if and only if Player **II** has a winning strategy in $G_L^\kappa(A, B)$, the **Lipschitz game of length κ for A, B** . It is a zero-sum, perfect information game of length κ , in which at each inning $\alpha < \kappa$ the two-players **I** and **II**

play $x_\alpha, y_\alpha \in 2$ with **I** playing first and at limit levels:

I	x_0	x_1	\cdots	x_α	\cdots
II	y_0	y_1	\cdots	y_α	\cdots

Player **II** wins just in case

$$\langle x_\alpha \mid \alpha < \kappa \rangle \in A \Leftrightarrow \langle y_\alpha \mid \alpha < \kappa \rangle \in B.$$

Similarly, we say that A is **continuously reducible** or **Wadge reducible**¹³ to B , in symbols $A \leq_W^\kappa B$, if there is a continuous $f: {}^\kappa 2 \rightarrow {}^\kappa 2$ such that $A = f^{-1}(B)$. Equivalently: $A \leq_W^\kappa B$ if and only if Player **II** has a winning strategy in $G_W^\kappa(A, B)$, the **Wadge game of length κ for A, B** . This game is similar to $G_L^\kappa(A, B)$, except that **II** can pass at any round, provided that at the end a sequence of length κ is produced.

A set $A \subseteq {}^\kappa 2$ is **\leq_L^κ -hard for a collection $\mathcal{S} \subseteq \mathcal{P}({}^\kappa 2)$** if $\forall B \in \mathcal{S} (B \leq_L^\kappa A)$. A set which is \leq_L^κ -hard for \mathcal{S} and moreover belongs to \mathcal{S} is said to be **\leq_L^κ -complete for \mathcal{S}** .

PROPOSITION 3.9. *Suppose $\mathcal{S} \subseteq \mathcal{P}({}^\kappa 2)$ and endow ${}^\kappa 2$ with the bounded topology.*

- (a) *If there is $A \subseteq {}^\kappa 2$ which is \leq_L^κ -hard for \mathcal{S} , then $\mathcal{S} \neq \mathcal{P}({}^\kappa 2)$.*
- (b) *Suppose \mathcal{S} is closed under complements and continuous preimages, i.e. $A \in \mathcal{S}$ implies $f^{-1}(A) \in \mathcal{S}$ for all continuous $f: {}^\kappa 2 \rightarrow {}^\kappa 2$. Then there is no \leq_L^κ -complete set for \mathcal{S} .*

PROOF. (a) If ${}^\kappa 2 \setminus A \not\leq_L^\kappa A$, then ${}^\kappa 2 \setminus A$ already witnesses that $\mathcal{S} \neq \mathcal{P}({}^\kappa 2)$, so we may assume that ${}^\kappa 2 \setminus A \leq_L^\kappa A$. Let $g: {}^\kappa 2 \rightarrow {}^\kappa 2$ be the continuous map defined by $g(x)(\alpha) := x(\alpha + 1)$ for all $\alpha < \kappa$, and let $\bar{A} := g^{-1}(A)$. Player **I** has a winning strategy in $G_L^\kappa(\bar{A}, A)$ by playing 0 (or 1 for that matter) at round 0 and at all limit rounds, and by following σ at all successor rounds, where σ is a winning strategy for **II** in $G_L^\kappa(A, {}^\kappa 2 \setminus A)$. Therefore $\bar{A} \not\leq_L^\kappa A$, and hence $\bar{A} \notin \mathcal{S}$.

(b) Towards a contradiction, suppose $A \in \mathcal{S}$ is \leq_L^κ -complete for \mathcal{S} . Since ${}^\kappa 2 \setminus A \in \mathcal{S}$ by closure under complements, then ${}^\kappa 2 \setminus A \leq_L^\kappa A$. Then the set \bar{A} defined as in part (a) contradicts the choice of A : in fact, the function g witnesses $\bar{A} \leq_W^\kappa A$, so that $\bar{A} \in \mathcal{S}$, but $\bar{A} \not\leq_L^\kappa A$. \square

3.2.2. *Properties of τ_p , τ_λ and τ_b .* For the space ${}^\omega 2$ the cardinality of the topology equals the cardinality of the continuum

$$|\tau_p({}^\omega 2)| = |\tau_b({}^\omega 2)| = |{}^\omega 2|.$$

The situation for ${}^\kappa 2$ when $\kappa > \omega$ is quite different. First of all note that the following trivial facts:

$$(3.2) \quad \tau_p \subseteq \tau_\lambda \wedge \tau_p \subseteq \tau_b,$$

and

$$(3.3) \quad \text{if } \mathcal{B} \text{ is a basis for a topology } \tau, \text{ then } |\tau| \leq |\mathcal{P}(\mathcal{B})|,$$

This is an immediate consequence of the fact that $\tau \mapsto \mathcal{P}(\mathcal{B})$, $U \mapsto \{B \in \mathcal{B} \mid B \subseteq U\}$, is injective.

LEMMA 3.10. *Let $\lambda < \kappa$ be uncountable cardinals, and let $\tau_p := \tau_p({}^\kappa 2)$, $\tau_\lambda := \tau_\lambda({}^\kappa 2)$, and $\tau_b := \tau_b({}^\kappa 2)$.*

- (a) $|\tau_p| = |\mathcal{P}(\kappa)|$.
- (b) $\mathcal{P}({}^\omega 2) \mapsto \tau_\lambda$ and $\mathcal{P}({}^\omega 2) \mapsto \tau_b$ and therefore $\tau_\lambda \twoheadrightarrow \mathcal{P}({}^\omega 2)$ and $\tau_b \twoheadrightarrow \mathcal{P}({}^\omega 2)$. Moreover $|\mathcal{P}({}^{<\lambda} 2)| \leq |\tau_\lambda| \leq |\mathcal{P}([\kappa]^{<\lambda})|$ and $|\tau_b| = |\mathcal{P}({}^{<\kappa} 2)|$.

¹³W.W. Wadge initiated the use of games to study continuous reducibility on the Baire space in [Wad83].

- (c) Assume AD. If κ is a surjective image of \mathbb{R} (i.e. $\kappa < \Theta$, where Θ is as in Definition 4.3), then $|\tau_p| < |\tau_\lambda|$ and $|\tau_p| < |\tau_b|$.
- (d) Assume AC. Then
- $2^\kappa < 2^{(2^{<\lambda})} \Rightarrow |\tau_p| < |\tau_\lambda|$,
 - $2^\kappa < 2^{(2^{<\kappa})} \Leftrightarrow |\tau_p| < |\tau_b|$,
 - $2^{(\kappa^{<\lambda})} < 2^{(2^{<\kappa})} \wedge \lambda < \min(\text{cof}(\kappa)^+, \kappa) \Rightarrow |\tau_\lambda| < |\tau_b|$.

PROOF. (a) Since $\mathcal{B}_p = \{\mathbf{N}_s \mid s \in \mathbf{Fn}(\kappa, 2; \omega)\}$ is a basis for τ_p and $|\mathbf{Fn}(\kappa, 2; \omega)| = \kappa$, then $|\tau_p| \leq |\mathcal{P}(\kappa)|$ by (3.3). For the other inequality use the injection $\mathcal{P}(\kappa) \rightarrow \tau_p$, $A \mapsto \bigcup_{\alpha \in A} \tilde{\mathbf{N}}_{\alpha,1}$.

(b) The map $\mathcal{P}(\omega 2) \rightarrow \mathcal{P}(\kappa 2)$, $\omega 2 \supseteq A \mapsto \bigcup_{s \in A} \mathbf{N}_s$, witnesses that $|\mathcal{P}(\omega 2)| \leq |\tau_\lambda|$ and $|\mathcal{P}(\omega 2)| \leq |\tau_b|$.

The function $\mathbf{Fn}(\kappa, 2; \lambda) \rightarrow [\kappa]^{<\lambda}$ defined by

$$s \mapsto \{2\alpha \mid \alpha \in \text{dom } s \wedge s(\alpha) = 0\} \cup \{2\alpha + 1 \mid \alpha \in \text{dom } s \wedge s(\alpha) = 1\}$$

is a bijection. Thus $[[\kappa]^{<\lambda}] = |\mathbf{Fn}(\kappa, 2; \lambda)| = |\mathcal{B}_\lambda|$, and hence $|\tau_\lambda| \leq |\mathcal{P}([\kappa]^{<\lambda})|$ by (3.3). Similarly, as $|\mathcal{B}_b| = |^{<\kappa}2|$, then $|\tau_b| \leq |\mathcal{P}(<\kappa 2)|$. In order to prove the other inequalities, consider the map

$$(3.4) \quad U: {}^{<\kappa}2 \rightarrow \mathcal{B}_b \subseteq \tau_b, \quad s \mapsto \mathbf{N}_{0(\text{lh } s) \smallfrown 1 \smallfrown s},$$

and notice that if $s \in {}^{<\lambda}2$, then $U(s) \in \mathcal{B}_\lambda \subseteq \tau_\lambda$. Then the injection $\mathcal{P}(<\kappa 2) \rightarrow \tau_b$, $A \mapsto \bigcup_{s \in A} U(s)$, witnesses $|\mathcal{P}(<\kappa 2)| \leq |\tau_b|$, while its restriction to $\mathcal{P}(<\lambda 2)$ has range contained in τ_λ and witnesses $|\mathcal{P}(<\lambda 2)| \leq |\tau_\lambda|$.

(c) By a standard result in determinacy (see Theorem 4.4 below), there is a surjection of $\omega 2$ onto $\mathcal{P}(\kappa)$, and hence $\omega 2 \twoheadrightarrow \tau_p$ by (a). Let τ be either τ_λ or τ_b , so that $|\tau_p| \leq |\tau|$ by (3.2). If $|\tau_p| = |\tau|$ then $\omega 2 \twoheadrightarrow \tau$, and therefore $\omega 2 \twoheadrightarrow \mathcal{P}(\omega 2)$ by (b), a contradiction. Therefore $|\tau_p| < |\tau|$.

(d) follows from (a) and (b). \square

REMARK 3.11. Assume AD and let κ be such that $\mathbb{R} \twoheadrightarrow \kappa$. Although \mathbb{R} does not surject onto $\tau_b(\kappa 2)$ by Lemma 3.10(b), at least we have that $\mathbb{R} \twoheadrightarrow \mathcal{B}_b(\kappa 2)$. This is because $|\mathcal{B}_b(\kappa 2)| = |^{<\kappa}2|$, $\kappa 2 \twoheadrightarrow {}^{<\kappa}2$, and $\mathbb{R} \twoheadrightarrow \kappa 2$ by Theorem 4.4. As $[\kappa]^{<\lambda} \rightarrow {}^{<\kappa}2$, this implies that $\mathbb{R} \twoheadrightarrow \mathcal{B}_\lambda(\kappa 2)$ for all $\omega < \lambda < \min(\text{cof}(\kappa)^+, \kappa)$, as well.

Let us summarize some properties of the topologies τ_p , τ_λ and τ_b on $\kappa 2$.

PROPOSITION 3.12. *Let $\lambda < \kappa$ be uncountable cardinals, and consider the space $\kappa 2$ with one of the topologies $\tau_p, \tau_\lambda, \tau_b$.*

- (a) *The topologies $\tau_p, \tau_\lambda, \tau_b$ are perfect, regular Hausdorff, and zero-dimensional.*
- (b) *τ_p is compact, while τ_λ and τ_b are not.*
- (c) *Let $\mathcal{U}_p, \mathcal{U}_\lambda$, and \mathcal{U}_b be open neighborhood bases of some $x \in \kappa 2$ with respect to τ_p, τ_λ , and τ_b . Then $\mathcal{U}_p \twoheadrightarrow \kappa$, $\mathcal{U}_b \twoheadrightarrow \text{cof}(\kappa)$, and (assuming AC) $\mathcal{U}_\lambda \twoheadrightarrow \kappa$; moreover there are \mathcal{U}_p and \mathcal{U}_b as above such that $|\mathcal{U}_p| = \kappa$ and $|\mathcal{U}_b| = \text{cof}(\kappa)$. Therefore τ_p and (assuming AC) τ_λ are never first countable, while τ_b is first countable if and only if $\text{cof}(\kappa) = \omega$.*
- (d) *Let $\mathcal{B}'_p, \mathcal{B}'_\lambda$ and \mathcal{B}'_b be bases for the topologies τ_p, τ_λ and τ_b , respectively, on $\kappa 2$. Then $\mathcal{B}'_p \twoheadrightarrow \kappa$, $\mathcal{B}'_\lambda \twoheadrightarrow {}^{<\lambda}2$ (and, assuming AC, also $\mathcal{B}'_\lambda \twoheadrightarrow \kappa$), and $\mathcal{B}'_b \twoheadrightarrow {}^{<\kappa}2$. Therefore the topologies τ_p, τ_λ and τ_b are never second countable.*
- (e) *The topology τ_p is separable if and only if $\kappa \leq |\omega 2|$. Therefore under AD the topology τ_p is never separable, while under AC it is separable if and only if $\kappa \leq 2^{\aleph_0}$.*
- (f) *If $D \subseteq \kappa 2$ is τ_λ -dense, then $D \twoheadrightarrow {}^{<\lambda}2$, hence τ_λ is never separable. Moreover there is a τ_λ -dense set of size $[[\kappa]^{<\lambda}]$. Thus assuming AC we get that the density character of τ_λ is between $2^{<\lambda}$ and $\kappa^{<\lambda}$. Further assuming that λ is inaccessible, one has that τ_λ has density λ if and only if $\kappa \leq 2^\lambda$.*

- (g) If $D \subseteq {}^\kappa 2$ is τ_b -dense, then $D \twoheadrightarrow {}^{<\kappa} 2$. Moreover there is a τ_b -dense set of size $|{}^{<\kappa} 2|$. In particular τ_b is never separable, and under AC it has density character $2^{<\kappa}$.
- (h) The topology τ_b is completely metrizable if and only if it is metrizable if and only if $\text{cof}(\kappa) = \omega$; the topologies τ_p and (assuming AC) τ_λ are never metrizable.
- (i) The topology τ_* with $*$ in $\{p, \lambda, b\}$ is closed under intersections of length $\leq \alpha$ (for some ordinal α) if and only if: $\alpha < \omega$ when $*$ = p , $\alpha < \lambda$ when $*$ = λ (assuming AC), $|\alpha| < \text{cof}(\kappa)$ when $*$ = b . In particular (3.1) holds.
- (j) Let $\mathcal{C}_p, \mathcal{C}_\lambda, \mathcal{C}_b$ be the collections of all clopen subsets of ${}^\kappa 2$ with respect to τ_p, τ_λ , and τ_b respectively. Then \mathcal{C}_p is a ω -algebra, that is an algebra of sets; \mathcal{C}_λ is a λ -algebra (assuming AC); \mathcal{C}_b is a $\text{cof}(\kappa)$ -algebra.

PROOF. Part (a) is trivial.

(b) Tychonoff's theorem for $({}^\kappa 2, \tau_p)$ is provable in ZF (see e.g. [Ker00, Theorem 4]), and $\{N_s \mid s \in {}^\omega 2\}$ is an infinite clopen partition of $({}^\kappa 2, \tau_\lambda)$ and $({}^\kappa 2, \tau_b)$.

(c) Fix $x \in {}^\kappa 2$, and let \mathcal{U} be one of $\mathcal{U}_p, \mathcal{U}_\lambda, \mathcal{U}_b$. We construct a map sending each $U \in \mathcal{U}$ to some $s(U) \subseteq x$ such that

$$(3.5) \quad N_{s(U)} \subseteq U,$$

and consider the neighborhood base $\mathcal{U}' = \{N_{s(U)} \mid U \in \mathcal{U}\}$. As $\mathcal{U} \twoheadrightarrow \mathcal{U}'$, it is enough to show that there is a map from \mathcal{U}' onto κ or $\text{cof}(\kappa)$.

Case $\mathcal{U} = \mathcal{U}_p$: choose $v(U) \in [\kappa]^{<\omega}$ such that (3.5) holds when $s(U) = x \upharpoonright v(U)$. The axiom of choice is not needed here, as $[\kappa]^{<\omega}$ is well-orderable. If $\alpha < \kappa$ and $U \in \mathcal{U}$ is contained in $\tilde{N}_{\alpha, x(\alpha)}$, then $\alpha \in v(U)$; this implies that $\kappa = \bigcup \{v(U) \mid U \in \mathcal{U}_p\}$, and therefore $\mathcal{U}' \twoheadrightarrow \kappa$.

Case $\mathcal{U} = \mathcal{U}_\lambda$: choose $s(U) \in \mathbf{Fn}(\kappa, 2; \lambda)$ so that (3.5) holds, and let $S = \{\text{dom } s(U) \mid U \in \mathcal{U}\}$. If $\alpha < \kappa$ and $U \in \mathcal{U}$ is contained in $\tilde{N}_{\alpha, x(\alpha)}$, then $\alpha \in \text{dom } s(U)$ and hence $\kappa = \bigcup S$. As every element of S has size $< \lambda$, then $\kappa \leq \lambda \cdot |S|$, so $\kappa \leq |S|$. Thus $\mathcal{U}' \twoheadrightarrow \kappa$.

Case $\mathcal{U} = \mathcal{U}_b$: choose $\alpha(U) \in \kappa$ such that (3.5) is satisfied when $s(U) = x \upharpoonright \alpha(U)$. If $\text{cof}(\kappa) \not\leq |\mathcal{U}'|$, then $\alpha := \sup_{U \in \mathcal{U}_b} \alpha(U) + 1 < \kappa$, so the open neighborhood $N_{x \upharpoonright \alpha}$ does not contain any element of the open neighborhood basis \mathcal{U}' , a contradiction.

The τ_p -neighborhood base $\{N_s \in \mathcal{B}_p \mid s \subseteq x\}$ has size κ , and the τ_b -neighborhood base $\{N_{x \upharpoonright \alpha_i} \in \mathcal{B}_b \mid i < \text{cof}(\kappa)\}$ has size $\text{cof}(\kappa)$, where the α_i are cofinal in κ .

The fact that $({}^\kappa 2, \tau_b)$ is first countable when $\text{cof}(\kappa) = \omega$ is immediate.

(d) Fix $x \in {}^\kappa 2$. The set $\mathcal{U}_p = \{U \in \mathcal{B}'_p \mid x \in U\}$ is a τ_p -neighborhood base of x , so $\mathcal{U}_p \twoheadrightarrow \kappa$ by part (c), whence $\mathcal{B}'_p \twoheadrightarrow \kappa$. A similar argument shows that $\mathcal{B}'_\lambda \twoheadrightarrow \kappa$ when assuming AC.

The map U defined in (3.4) has the property that $U(s) \cap U(t) = \emptyset$ for $s \neq t$. Therefore the map $\mathcal{B}'_b \rightarrow {}^{<\kappa} 2$ defined by

$$B \mapsto \begin{cases} \emptyset & \text{if } \forall s \in {}^{<\kappa} 2 (B \not\subseteq U(s)), \\ s & \text{if } B \subseteq U(s), \end{cases}$$

is a well-defined surjection, and similarly for $\mathcal{B}'_\lambda \twoheadrightarrow {}^{<\lambda} 2$.

(e) We first show that ${}^I 2$ with the product topology is separable, when $I \subseteq {}^\omega 2$.¹⁴ The set

$$E_n := \{\varphi \in {}^I 2 \mid \forall f, g \in I (f \upharpoonright n = g \upharpoonright n \Rightarrow \varphi(f) = \varphi(g))\}$$

¹⁴This is a particular case of the more general statement that the product of κ -many separable spaces is separable if and only if $\kappa \leq 2^{\aleph_0}$ [Kun80, Exercises 3 and 4, page 86]. The reason for including the proof here is to show that it holds in ZF.

is countable since it is in bijection with the set of all functions from ${}^n 2$ to 2 , and hence

$$E := \bigcup_{n \in \omega} E_n$$

is countable. We now argue that E is dense by showing that it intersects every basic open set of ${}^I 2$ of the form

$$V := \{\varphi \in {}^I 2 \mid \varphi(f_1) = i_1 \wedge \dots \wedge \varphi(f_m) = i_m\}$$

with $f_1, \dots, f_m \in I$ and $i_1, \dots, i_m \in \{0, 1\}$. Let n be large enough so that $f_1 \upharpoonright n, \dots, f_m \upharpoonright n$ are all distinct, and let $\varphi \in E_n$ be such that $\varphi(f_j) = i_j$ ($j = 1, \dots, m$). Then $\varphi \in E \cap V$ as required.

Conversely, suppose $\kappa \not\leq |{}^\omega 2|$. Given any set $\{f_n \mid n \in \omega\} \subseteq {}^\kappa 2$ let $F: \kappa \rightarrow {}^\omega 2$ be defined by

$$F(\alpha)(n) := f_n(\alpha).$$

By case assumption F cannot be injective, and hence there are $\alpha < \beta < \kappa$ such that $F(\alpha) = F(\beta)$, that is $f_n(\alpha) = f_n(\beta)$ for all n . Thus $\{f_n \mid n \in \omega\}$ is disjoint from the basic open set $\{f \in {}^\kappa 2 \mid f(\alpha) = 0 \wedge f(\beta) = 1\}$ and therefore it is not dense.

The result under AD follows from the fact that ω_1 does not embed into ${}^\omega 2$.

(f) Let $D \subseteq {}^\kappa 2$ be τ_λ -dense. As the sets $U(s)$ defined in (3.4) are pairwise disjoint, we obtain a surjection $D \twoheadrightarrow {}^{<\lambda} 2$ by setting

$$(3.6) \quad x \mapsto \begin{cases} \emptyset & \text{if } x \notin \bigcup_{s \in {}^{<\lambda} 2} U(s) \\ s & \text{if } x \in U(s). \end{cases}$$

Moreover, the set $\{\chi_u \mid u \in [\kappa]^{<\lambda}\}$ is τ_λ -dense and has cardinality $|[\kappa]^{<\lambda}|$, where $\chi_u: \kappa \rightarrow \{0, 1\}$ is the characteristic function of u . Finally, the additional part concerning the case when λ is inaccessible (assuming AC) can be proved by replacing ω with λ in the proof of part (e).

(g) The argument is similar to that of part (f). If D is a dense set in $({}^\kappa 2, \tau_b)$, then one gets a surjection $D \twoheadrightarrow {}^{<\kappa} 2$ by replacing λ with κ in (3.6). Moreover, the set $\{s \cap 1 \cap 0^{(\kappa)} \mid s \in {}^{<\kappa} 2\}$ is τ_b -dense and has cardinality $|{}^{<\kappa} 2|$.

(h) Since a metric space is first countable, by part (c) it is enough to show that $({}^\kappa 2, \tau_b)$ is completely metrizable when $\text{cof}(\kappa) = \omega$. This easily follows from the application of the Birkhoff-Kakutani theorem [MZ55, §1.22, p. 34] to the first countable Hausdorff topological group $({}^\kappa 2, \tau_b)$ with the operation $(x, y) \mapsto x + y$ defined by $(x + y)(\alpha) := x(\alpha) + y(\alpha)$ modulo 2. For a more direct proof, let $\langle \lambda_n \mid n \in \omega \rangle$ be a strictly increasing sequence of ordinals cofinal in κ , and equip $X := \prod_{n \in \omega} {}^{\lambda_n} 2$ with the product of the discrete topologies on each ${}^{\lambda_n} 2$: then the metric d on X defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 2^{-n}$ with n smallest such that $x(n) \neq y(n)$ if $x \neq y$ is complete and compatible with the topology of X . Therefore, X and all its closed subsets are completely metrizable. Since the map ${}^\kappa 2 \rightarrow X$, $x \mapsto \langle x \upharpoonright \lambda_n \mid n \in \omega \rangle$ is a well-defined homeomorphism between $({}^\kappa 2, \tau_b)$ and a closed subset of X , we get the desired result.

(i) Let $\langle U_\beta \mid \beta < \alpha \rangle$ be a sequence sets in τ_λ , with $\alpha < \lambda$. It is enough to show that for each $x \in V := \bigcap_{\beta < \alpha} U_\beta$ there is $s \in \mathbf{Fn}(\kappa, 2; \lambda)$ such that $x \in N_s \subseteq V$. For each $\beta < \alpha$ choose $u_\beta \in [\kappa]^{<\lambda}$ such that $N_{x \upharpoonright u_\beta} \subseteq U_\beta$. Then $v := \bigcup_{\beta < \alpha} u_\beta \in [\kappa]^{<\lambda}$ and $N_{x \upharpoonright v} \subseteq V$. On the other hand, the set N_s with $s \in {}^\lambda 2$ witnesses that τ_λ is not closed under intersections of length λ , and similarly τ_p is not closed under infinite intersections.

The argument for τ_b is similar. Suppose first $|\alpha| < \text{cof}(\kappa)$ and let $\langle U_\beta \mid \beta < \alpha \rangle$ be a sequence of sets in τ_b , and let V be as before. For any $x \in \bigcap_{\beta < \alpha} U_\beta$ we construct an $s \in {}^{<\kappa} 2$ such that $N_s \subseteq V$. For each $\beta < \alpha$ let $f(\beta)$ be the smallest $\gamma < \kappa$ such that $N_{x \upharpoonright \gamma} \subseteq U_\beta$. By case assumption $\text{ran}(f)$ is bounded in κ , so $N_s \subseteq V$ with $s := x \upharpoonright \sup \text{ran}(f)$. Suppose now $|\alpha| \geq \text{cof}(\kappa)$. Let $\langle \gamma_\beta \mid \beta < \text{cof}(\kappa) \rangle$ be increasing and cofinal in κ , and let $V_\beta := N_{0 \upharpoonright \gamma_\beta}$ if $\beta < \text{cof}(\kappa)$ and $V_\beta := {}^\kappa 2$ if $\text{cof}(\kappa) \leq \beta < \alpha$. Then the set $\bigcap_{\beta < \alpha} V_\beta = \bigcap_{\beta < \text{cof}(\kappa)} V_\beta = \{0^{(\kappa)}\}$ is closed and not open.

(j) follows from (i). \square

REMARKS 3.13. (i) Even if $({}^\kappa 2, \tau_b)$ is never compact by (b) above, some form of compactness is available in certain cases: for example, as shown in [MR13, Theorem 5.6], $({}^\kappa 2, \tau_b)$ is κ -compact¹⁵ (i.e. such that every τ_b -open covering of ${}^\kappa 2$ has a subcovering of size $< \kappa$) if and only if κ is a weakly compact cardinal.

- (ii) By (c) and (d) of Proposition 3.12, the space $({}^\kappa 2, \tau_p)$ with $\omega < \kappa \leq |\omega 2|$ witnesses (in ZF) the well-known fact that separability does not imply second countability. Notice that the converse implication “a second countable space is separable” is equivalent to AC_ω : given nonempty sets A_n , endow $X := \bigcup_n A_n$ with the topology generated by the A_n ’s so that X is second countable. Then any function enumerating a dense subset of X yields a choice function for the A_n ’s. (The other direction of the equivalence is immediate.)
- (iii) Part (e) of Proposition 3.12 shows that the separability of $({}^{\omega 1} 2, \tau_p)$ is independent of $\text{ZF} + \text{DC}$.
- (iv) The proof of part (e) of Proposition 3.12 also shows that if A is arbitrary, X is a separable space, and ${}^A X$ is endowed with the product topology,

$${}^A X \text{ is separable} \Leftrightarrow |A| \leq |\omega 2|.$$

Thus the Perfect set Property PSP (which follows from AD) implies that the space ${}^A X$ is separable if and only if $|A| \leq \omega$ or $|A| = |\omega 2|$.

- (v) The techniques of classical descriptive set theory heavily rely on the existence of a (complete) metric, and hence by part (h) of Proposition 3.12 they cannot be directly applied to ${}^\kappa 2$ with τ_p , τ_λ , and τ_b when $\kappa > \omega$. In fact only a handful of “positive” results can be generalized to e.g. $({}^\kappa 2, \tau_b)$ (see [FHK14, MR13]).
- (vi) If $\text{cof}(\kappa) < \kappa$ then $\mathcal{C}_{\text{cof}(\kappa)}$ and \mathcal{C}_b are distinct $\text{cof}(\kappa)$ -algebras, since $\mathcal{B}_b \subseteq \mathcal{C}_b$ and by the proof of Lemma 3.4(c) there are sets in \mathcal{B}_b that are not $\tau_{\text{cof}(\kappa)}$ -open.

PROPOSITION 3.14. *Let $\lambda < \kappa$ be uncountable cardinals, and work in ${}^\kappa 2$. Assume $\text{AC}_\omega(\mathbb{R})$. There is a set which is τ_λ -open and τ_b -open, and it is not τ_p -Borel.*

Before starting the proof, let’s fix a bit of notation: if $\nu \leq \mu$ are infinite cardinals, the **inclusion map** $i: {}^\nu 2 \hookrightarrow {}^\mu 2$ is the identity if $\nu = \mu$, and it is the map $x \mapsto x \smallfrown 0^{(\mu)}$ if $\nu < \mu$.

PROOF. By $\text{AC}_\omega(\mathbb{R})$ there is a non-Borel set $A \subseteq {}^\omega 2$. Then $U := \bigcup_{s \in A} N_s^\kappa$ is a τ_λ -open subset of ${}^\kappa 2$. As the inclusion map $({}^\omega 2, \tau_p) \hookrightarrow ({}^\kappa 2, \tau_p)$ is continuous and since the generalized pointclass \mathbf{B} is boldface, then U cannot be Borel in $({}^\kappa 2, \tau_p)$. \square

Using the arguments above, one can prove the next result describing the continuous functions between generalized Cantor spaces.

PROPOSITION 3.15. *Let $\lambda \leq \text{cof}(\kappa)$ and $\lambda' \leq \text{cof}(\kappa')$ with $\kappa \leq \kappa'$ cardinals, and let $i: {}^\kappa 2 \hookrightarrow {}^{\kappa'} 2$ be the inclusion map. Then*

- (a) $i: ({}^\kappa 2, \tau_\lambda) \hookrightarrow ({}^{\kappa'} 2, \tau_{\lambda'})$ is continuous if and only if $\lambda' \leq \lambda$. Therefore i is continuous when ${}^{\kappa'} 2$ is topologized with τ_p and ${}^\kappa 2$ is topologized with any of $\tau_p, \tau_\lambda, \tau_b$.
- (b) If ${}^{\kappa'} 2$ is given the bounded topology, then i is continuous if and only if $\kappa = \kappa'$.
- (c) $i: ({}^\kappa 2, \tau_b) \hookrightarrow ({}^{\kappa'} 2, \tau_{\lambda'})$ is continuous if and only if $\lambda' \leq \text{cof}(\kappa)$.

4. Generalized Borel sets

4.1. Basic facts.

¹⁵In general topology, κ -compact spaces are also called κ -Lindelöf.

4.1.1. *α -Borel sets.* The following definition introduces a natural generalization of the notion of Borel subset of a topological space. It plays a central role in the analysis of models of AD, but it is also of primary interest in other areas of set theory and in general topology.

DEFINITION 4.1. If $X = (X, \tau)$ is a topological space and α an ordinal, the collection of **α -Borel sets (with respect to τ)** is

$$\mathbf{B}_\alpha(X, \tau) := \text{Alg}(\tau, \alpha),$$

the smallest family of subsets of X containing all τ -open sets and closed under the operations of complementation and well-ordered unions of length $< \alpha$. A set is **∞ -Borel (with respect to τ)** if it is α -Borel (with respect to τ) for some α , i.e. if it belongs to $\mathbf{B}_\infty(X, \tau) := \bigcup_{\alpha \in \text{Ord}} \mathbf{B}_\alpha(X, \tau)$. As usual when drop the reference to X and/or τ when there is no danger of confusion.

Using the stratification of $\text{Alg}(\tau, \alpha)$ in terms of the subcollections $\Sigma_\gamma(\tau, \alpha)$ described in Section 2.4, it is easy to check that the operation \mathbf{B}_α assigning to each nonempty topological space (X, τ) the collection $\mathbf{B}_\alpha(X, \tau)$ of its α -Borel subsets is a hereditary general boldface pointclass.

- REMARKS 4.2. (i) If $\alpha \leq \alpha'$ and $\tau \subseteq \tau'$ then $\mathbf{B}_\alpha(X, \tau) \subseteq \mathbf{B}_{\alpha'}(X, \tau')$.
(ii) Both expressions $\mathbf{B}_{\alpha+1}(X, \tau)$ and $\mathbf{B}_{\alpha^+}(X, \tau)$ denote the same collection of subsets of X , but the former is often preferable when discussing models of set theory, since the term $\alpha+1$ is absolute, while α^+ is not.
(iii) $\mathbf{B}_{\omega_1}(X, \tau) = \mathbf{B}_{\omega+1}(X, \tau)$ is the usual collection of Borel subsets of $X = (X, \tau)$.

The notion of α -Borel set can be trivial under choice. For example any subset of an Hausdorff space X is in $\mathbf{B}_\alpha(X)$ if $\alpha > |X|$, and therefore AC implies that $\mathbf{B}_\infty(X) = \mathcal{P}(X)$. On the other hand, Theorem 4.18 below shows that $\mathbf{B}_{\kappa+1}(\kappa^2, \tau_b) \neq \mathcal{P}(\kappa^2)$ holds in ZFC, i.e. that $\mathbf{B}_{\kappa+1}(\kappa^2, \tau_b)$ is never a trivial class.

The situation in the AD-world is more subtle: $\mathbf{B}_\infty(\omega^2) = \mathcal{P}(\omega^2)$ holds in every known model of AD, and in fact the received opinion is that this must always be the case — this is one-half of the well-known conjecture that $\text{AD} \Rightarrow \text{AD}^+$ (see Section 9.4). Recall from Section 2.5.3 that the length of a prewellordering \preceq is the length of the associated regular norm; this ordinal is denoted by $\text{lh}(\preceq)$.

DEFINITION 4.3. $\Theta := \sup\{\alpha \in \text{Ord} \mid \text{there exists a surjection } f: \mathbb{R} \rightarrow \alpha\}$. Equivalently

$$\Theta := \sup\{\text{lh}(\preceq) \mid \preceq \text{ is a prewellordering of } \mathbb{R}\},$$

and \mathbb{R} can be replaced by any uncountable Polish space such as ${}^\omega 2$ or ${}^\omega \omega$.

It is an easy exercise to show (in ZF) that Θ is a cardinal, while using choice $\Theta = (2^{\aleph_0})^+$. In contrast, the following theorem of H. Friedman shows that Θ is a limit cardinal under AD (see [Kan03, Theorem 28.15] or [Mos09, Exercise 7D.19] for a proof).

THEOREM 4.4 (AD). *If $\lambda < \Theta$, then $\mathbb{R} \rightarrow \mathcal{P}(\lambda)$. In particular, $\lambda^+ < \Theta$ for all $\lambda < \Theta$.*

In fact, AD implies that Θ is quite large (e.g. larger than the first fixed point of the \aleph -sequence), it has lot of measurable cardinals below it, and so on. For a proof of these results see e.g. [Mos09] and the references contained therein.

Under AD it is no longer true that successor cardinals are regular — see Section 4.2. The next result shows that the regular cardinals are cofinal in Θ .

LEMMA 4.5. *Assume AD. For all $\alpha < \Theta$ there is a regular cardinal $\alpha \leq \lambda < \Theta$.*

SKETCH OF THE PROOF. Let $\Gamma := \text{pos } \Sigma_1^1(\preceq)$ be defined as in [Mos09, Section 7C], where \preceq is a prewellordering of ${}^\omega\omega$ of length α . Then Γ is ω -parametrized, so ${}^\omega\omega \rightarrow \Gamma$ where Γ is the boldface version of Γ as defined in [Mos09, Section 3H]. Since Γ satisfies the hypotheses of [Mos09, Theorem 7D.8], then

$\lambda := \sup \{\beta \in \text{Ord} \mid \beta \text{ is the length of a strict well-founded relation in } \Gamma \text{ with field in } {}^\omega\omega\}$ is a regular cardinal. Notice that $\lambda < \Theta$, and that $\alpha \leq \lambda$ by our choice of \preceq . \square

PROPOSITION 4.6 (AD). *If $\kappa < \Theta$ is an infinite cardinal, then $\mathbb{R} \rightarrow \mathbf{B}_{\kappa+1}({}^\omega 2)$, and hence $\mathbf{B}_{\kappa+1}({}^\omega 2) \neq \mathcal{P}({}^\omega 2)$.*

PROOF. By Lemma 4.5 let $\kappa < \lambda < \Theta$ be regular. By Lemma 2.3 ${}^{<\lambda}({}^\omega 2) \rightarrow \mathbf{B}_{\kappa+1}({}^\omega 2)$, so it is enough to show that ${}^\omega 2 \rightarrow {}^{<\lambda}({}^\omega 2)$. This follows from ${}^{<\lambda}({}^\omega 2) \rightarrow {}^{\lambda \times \omega} 2 \rightarrow {}^\lambda 2$ and Theorem 4.4. \square

4.1.2. *Borel codes.* Let $X = (X, \tau)$ be a topological space and let \mathcal{S} be a basis for τ . An α -**Borel code** is a pair $C = (T, \phi)$ where T is a well-founded descriptive set-theoretic tree (of height $\leq \omega$) on some $\beta < \alpha$ and $\phi: T \rightarrow \mathcal{P}(X)$ is a map such that

$$(4.1a) \quad \phi(t) \in \tau, \text{ if } t \text{ is a terminal node of } T,$$

$$(4.1b) \quad \phi(t) = X \setminus \bigcap \{\phi(s) \mid s \text{ is an immediate successor of } t \text{ in } T\}, \text{ otherwise.}$$

The code C canonically determines a set $\phi(\emptyset) \in \mathbf{B}_\alpha(X, \tau)$, called the α -**Borel set coded by C** . A set coded by some C is called an **effective α -Borel set**, and

$$\mathbf{B}_\alpha^e(X, \tau)$$

is the collection of all these sets. If condition (4.1a) is strengthened to

$$(4.1c) \quad \phi(t) \in \mathcal{S}, \text{ if } t \text{ is a terminal node of } T,$$

we say that the code (T, ϕ) takes values in \mathcal{S} , and write

$$\mathbf{B}_\alpha^e(X, \mathcal{S})$$

for the collection of sets admitting such a code. (As usual, we drop X, τ or \mathcal{S} from our notation when they are clear from the context.)

REMARKS 4.7. (i) The relation $x \in \phi(C)$ is absolute for transitive models of enough set theory containing x and C , thus “effective Borelness” is a more robust notion than that of “Borelness”. Clearly

$$\mathbf{B}_\alpha^e(X, \tau) \subseteq \mathbf{B}_\alpha(X, \tau),$$

and the converse inclusion holds under AC. On the other hand, if choice fails badly the two notions can be quite different — $\mathbf{B}_\alpha^e(\mathbb{R})$ is always a surjective image of \mathbb{R} if α is countable,¹⁶ but in the Feferman-Lévy model (where \mathbb{R} is a countable union of countable sets) it is true that $\mathbf{B}_{\omega+1}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$.

(ii) It is immediate that $\mathbf{B}_\alpha^e(X, \mathcal{S}) \subseteq \mathbf{B}_\alpha^e(X, \tau)$, but the reverse inclusion need not hold, even if AC is assumed. The problem is that $\mathbf{B}_\alpha^e(X, \mathcal{S}) \subseteq \text{Alg}(\mathcal{S}, \alpha)$ (and in fact equality holds under AC), and in order to show that $\tau \subseteq \text{Alg}(\mathcal{S}, \alpha)$, one needs some further assumptions on the size of a basis of τ . For example, if $(X, \tau) := ({}^\kappa 2, \tau_p)$ and $\mathcal{S} := \mathcal{B}_p$, then ZF proves $|\mathcal{B}_p| = \kappa$ and

$$\mathbf{B}_{\kappa+1}^e(\mathcal{B}_p) = \mathbf{B}_{\kappa+1}^e(\tau_p).$$

¹⁶Every effective $\omega + 1$ -Borel code for a subset of a second countable topological space is obtained from a tree on ω , and hence, using the bijection between ${}^{<\omega}\omega$ and ω , its characteristic function can be identified with an element of ${}^\omega 2$.

On the other hand, if $(X, \tau) := (\kappa 2, \tau_b)$ and $\mathcal{S} := \mathcal{B}_b$, then working in $\text{ZFC} + 2^{(2^{<\kappa})} > 2^\kappa$ we have that $|\tau_b| = 2^{(2^{<\kappa})}$ by Lemma 3.10(b) and $|\text{Alg}(\mathcal{B}_b, \kappa + 1)| = 2^\kappa$ by $|\mathcal{B}_b| = 2^{<\kappa}$ and Lemma 2.3, whence $\tau_b \not\subseteq \text{Alg}(\mathcal{B}_b, \kappa + 1)$ and

$$\mathbf{B}_{\kappa+1}(\tau_b) = \mathbf{B}_{\kappa+1}^e(\tau_b) \neq \mathbf{B}_{\kappa+1}(\mathcal{B}_b).$$

Similarly, if $(X, \tau) := (\kappa 2, \tau_\lambda)$ with $\omega < \lambda \leq \text{cof}(\kappa)$ and $\mathcal{S} := \mathcal{B}_\lambda$, then assuming $\text{ZFC} + 2^{(2^{<\lambda})} > 2^\kappa$ we have that $\tau_\lambda \not\subseteq \text{Alg}(\mathcal{B}_\lambda, \kappa + 1)$ and thus $\mathbf{B}_{\kappa+1}(\tau_\lambda) = \mathbf{B}_{\kappa+1}^e(\tau_\lambda) \neq \mathbf{B}_{\kappa+1}(\mathcal{B}_\lambda)$.

The notion of $\alpha + 1$ -Borel set can be extended to the case when the ordinal α is replaced by an arbitrary set.

DEFINITION 4.8. Let $X = (X, \tau)$ be a topological space and let J be an arbitrary set.

- (i) $\mathbf{B}_{J+1}(X, \tau)$ is the smallest collection of subsets of X , containing all open sets, and closed under complements and unions of the form $\bigcup_{j \in J} Y_j$ with $\{Y_j \mid j \in J\} \subseteq \mathbf{B}_{J+1}(X)$.
- (ii) A code (T, ϕ) for $A \in \mathbf{B}_{J+1}(X, \tau)$ is a well-founded descriptive set-theoretic tree T (of height $\leq \omega$) on J together with a map ϕ with domain T satisfying (4.1a)–(4.1b), and such that $\phi(\emptyset) = A$. The family of all $A \in \mathbf{B}_{J+1}(X, \tau)$ which admit a code is denoted with $\mathbf{B}_{J+1}^e(X, \tau)$, and if the codes satisfy (4.1c), we write $\mathbf{B}_{J+1}^e(X, \mathcal{S})$.

Note that $\mathbf{B}_{J+1}(X)$ and $\mathbf{B}_{J+1}^e(X)$ are closed under unions of the form $\bigcup_{j' \in J'} Y_{j'}$ where the set of indexes J' is the surjective image of J , and that

$$\mathbf{B}_{J+1}^e(X, \mathcal{S}) \subseteq \mathbf{B}_{J+1}^e(X, \tau) \subseteq \mathbf{B}_{J+1}(X, \tau).$$

LEMMA 4.9. Let $J := {}^{<\kappa}2$.

- (a) $\mathbf{B}_{J+1}^e(\kappa 2, \tau_p) = \mathbf{B}_{J+1}^e(\kappa 2, \tau_b)$ and $\mathbf{B}_{J+1}(\kappa 2, \tau_p) = \mathbf{B}_{J+1}(\kappa 2, \tau_b)$.
- (b) $\mathbf{B}_{\kappa+1}^e(\kappa 2, \tau_b) \subseteq \mathbf{B}_{J+1}^e(\kappa 2, \tau_p)$ and $\mathbf{B}_{\kappa+1}(\kappa 2, \tau_b) \subseteq \mathbf{B}_{J+1}(\kappa 2, \tau_p)$.

PROOF. (a) Any τ_b -basic open set N_s is the intersection of a $<\kappa$ -sequence of τ_p -basic open sets, in particular it is effective $\kappa + 1$ -Borel (thus also effective $<\kappa 2 + 1$ -Borel, as $<\kappa 2 \twoheadrightarrow \kappa$) with respect to τ_p . If U is τ_b -open, then $U = \bigcup_{s \in S} N_s$ where $S := \{s \in {}^{<\kappa}2 \mid N_s \subseteq U\}$, and from S a code witnessing that U is effectively $<\kappa 2 + 1$ -Borel with respect to τ_p can be constructed, without any appeal to choice principles. Therefore every (effective) $<\kappa 2 + 1$ -Borel set in the topology τ_b is (effective) $<\kappa 2 + 1$ -Borel in the topology τ_p . The other inclusion follows from $\tau_p \subseteq \tau_b$.

(b) follows from (a) and $J = {}^{<\kappa}2 \twoheadrightarrow \kappa$. \square

COROLLARY 4.10 (AC). If $2^{<\kappa} = \kappa$, then $\mathbf{B}_{\kappa+1}(\kappa 2, \tau_p) = \mathbf{B}_{\kappa+1}(\kappa 2, \tau_b)$.

Notice that Corollary 4.10 applies to any cardinal (not just the regular ones). In particular in models of GCH the $\kappa + 1$ -Borel sets of $\kappa 2$ with respect to the topologies τ_p , τ_λ , and τ_b are the same. This should be contrasted with the results on ${}^\kappa\kappa$ — see Remark 6.3(iii).

4.2. Intermezzo: the projective ordinals. Proposition 4.6 shows that under AD each general boldface pointclass \mathbf{B}_α is proper, as long as $\alpha < \Theta$. If α is an odd projective ordinal then $\mathbf{B}_\alpha(\omega 2)$ can be pinned-down in the projective hierarchy.

DEFINITION 4.11. Given any boldface pointclass $\mathbf{\Gamma}$, let

$$\delta_\mathbf{\Gamma} := \sup \{ \text{lh}(\preceq) \mid \preceq \text{ is a prewellordering of } \mathbb{R} \text{ in } \mathbf{\Delta}_\mathbf{\Gamma} \}.$$

For ease of notation set

$$\delta_n^1 := \delta_{\Sigma_n^1} \quad \text{and} \quad \delta_1^2 := \delta_{\Sigma_1^2}.$$

The δ_n^1 are called **projective ordinals**.

In the definition above, the set \mathbb{R} can be replaced by any other uncountable Polish space, such as ${}^\omega 2$ or ${}^\omega \omega$, and the pointclass Γ could be replaced with its dual $\check{\Gamma}$, i.e. $\delta_\Gamma = \delta_{\check{\Gamma}}$. In particular $\delta_n^1 = \delta_{\Pi_n^1}$ and $\delta_1^2 = \delta_{\Pi_1^2}$. Since the initial segments of the prewellordering induced by a Γ -norm are in Δ_Γ , we have the following

FACT 4.12. *The length of a Γ -norm on a set $A \in \Gamma({}^\omega \omega)$ is $\leq \delta_\Gamma$, and if the length is δ_Γ then $A \notin \Delta_\Gamma({}^\omega \omega)$.*

In other words: the ordinal δ_Γ cannot be attained by a Δ_Γ prewellordering of a set in Δ_Γ . For example: from $\text{ZF} + \text{AC}_\omega(\mathbb{R})$ it follows that $\delta_1^1 = \omega_1$, and hence every Π_1^1 -norm on ${}^\omega 2$ (equivalently: every Δ_1^1 prewellordering of ${}^\omega 2$) has countable length, but there are Π_1^1 -norms on (proper) Π_1^1 sets of length ω_1 .

The ordinal δ_Γ carries a lot of information on the nature of the pointclass Γ . In particular, the projective ordinals can be used to describe the projective pointclasses. For example Martin and Moschovakis proved that under $\text{AD} + \text{DC}$

$$(4.2) \quad \mathbf{B}_{\delta_{2n+1}^1}({}^\omega 2) = \Delta_{2n+1}^1.$$

By the late 70's it became clear that most of the structural problems on the projective pointclasses can be reduced to questions regarding the projective ordinals, and the computation (under AD) of the δ_n^1 in terms of the \aleph -function came to the fore as a crucial problem. Recall that $\delta_1^1 = \omega_1$. Martin proved that AD implies that $\delta_2^1 = \omega_2$, $\delta_3^1 = \aleph_{\omega+1}$, and $\delta_4^1 = \aleph_{\omega+2}$. By work of Kechris, Martin and Moschovakis (see [Kec78]), assuming AD :

- (A) all δ_n^1 's are regular cardinals,¹⁷ in fact measurable cardinals;
- (B) $\delta_{2n+2}^1 = (\delta_{2n+1}^1)^+$;
- (C) $\delta_{2n+1}^1 = (\lambda_{2n+1}^1)^+$, where λ_{2n+1}^1 is a cardinal of countable cofinality.

Recall that κ is measurable if there is a κ -complete non-principal ultrafilter on κ . The theory ZF proves that if κ is measurable, then it is regular (and therefore it is a cardinal), but under AD measurable cardinals need not to be limit cardinals, and successor cardinals need not to be regular: for example ω_1 and ω_2 are measurable, and for $n \geq 3$ the ω_n 's are singular of cofinality ω_2 . Working in $\text{ZFC} + \text{AD}^{L(\mathbb{R})}$, no $(\omega_n)^{L(\mathbb{R})}$ can be a cardinal for $n \geq 3$, neither can be $(\aleph_\omega)^{L(\mathbb{R})}$, and hence

$$(4.3) \quad \delta_3^1 = (\aleph_{\omega+1})^{L(\mathbb{R})} \leq \aleph_3.$$

By (B) and (C) the computation of the projective cardinals in models of determinacy boils down to determine the value of λ_{2n+1}^1 or, equivalently, of δ_{2n+1}^1 for $n \geq 2$. This was achieved by Jackson [Jac89] who proved the general formula

$$\lambda_{2n+1}^1 = \aleph_{\gamma(2n-1)}$$

where

$$\gamma(1) = \omega \quad \text{and} \quad \gamma(n+1) = \omega^{\gamma(n)}.$$

En route to proving these results, Jackson verified that every regular cardinal smaller than $\delta_\omega := \sup_n \delta_n^1 = \aleph_{\varepsilon_0}$ is measurable, and was able to compute exactly all these cardinals: between δ_{2n+1}^1 and δ_{2n+3}^1 there are $2^{n+1} - 1$ regular (in fact: measurable) cardinals. From this it follows that δ_{2n+1}^1 is the $(2^{n+1} - 1)$ -st uncountable regular cardinal. By these results, and arguing as for (4.3) we obtain:

COROLLARY 4.13. *Work in ZFC and suppose there is an inner model containing all reals and satisfying AD . Then $\delta_{2n+1}^1 \leq \aleph_{2^{n+1}-1}$.*

¹⁷Thus the δ_n^1 are also called **projective cardinals**.

REMARK 4.14. Since $\delta_n^1 < \Theta \leq (2^{\aleph_0})^+$, (4.3) and Corollary 4.13 become trivial if the continuum is small. On the other hand, if the continuum is large, then by (4.2) large cardinals imply that every projective set is in $\mathbf{B}_{\aleph_\omega}(\omega 2)$.

4.3. * More on generalized Borel sets.

4.3.1. *The (generalized) Borel hierarchy.* The collection $\mathbf{B}(X)$ of all Borel subsets of a topological space (X, τ) is stratified in a hierarchy by $\Sigma_\alpha^0(X)$ and $\Pi_\alpha^0(X)$. A similar result holds for generalized Borel sets: in fact $\mathbf{B}_{\kappa+1}(X)$ is the smallest $\kappa+1$ -algebra on X containing τ , so with the notation of Section 2.4, $\mathbf{B}_{\kappa+1}(X) = \bigcup_{\alpha \in \text{Ord}} \Sigma_\alpha(\tau, \kappa+1) = \bigcup_{\alpha \in \text{Ord}} \Pi_\alpha(\tau, \kappa+1)$. For the sake of uniformity of notation, when $X = {}^\kappa 2$ with the bounded topology, we write $\Sigma_\alpha^0(\tau_b)$ and $\Pi_\alpha^0(\tau_b)$ instead of $\Sigma_\alpha(\tau_b, \kappa+1)$ and $\Pi_\alpha(\tau_b, \kappa+1)$.

A standard result in classical descriptive set theory is that if X is an uncountable Polish space, then the Borel hierarchy $\langle \Sigma_\alpha^0(X) \mid 1 \leq \alpha < \omega_1 \rangle$ does not collapse,¹⁸ i.e. $\Sigma_\alpha^0(X) \subset \Sigma_\beta^0(X)$ for $1 \leq \alpha < \beta < \omega_1$. This follows from the fact that each $\Sigma_\alpha^0(\omega\omega)$ has a universal set (Section 2.5.2). Generalizing this notion, given a boldface pointclass Γ and a topological space X , a set $\mathcal{U} \in \Gamma(X \times X)$ is **universal for $\Gamma(X)$** if $\Gamma(X) = \{\mathcal{U}^{(y)} \mid y \in X\}$ where $\mathcal{U}^{(y)} := \{x \in X \mid (x, y) \in \mathcal{U}\}$. Corollary 4.16 below shows that for ${}^\kappa 2$ with the bounded topology, there may be no universal sets for $\Sigma_\alpha^0(\tau_b)$, but nevertheless the hierarchy does not collapse (Proposition 4.19).

LEMMA 4.15. *In the space $({}^\kappa 2, \tau_b)$, for every $1 \leq \alpha < \beta \in \text{Ord}$*

$$\Sigma_\alpha^0(\tau_b) \cup \Pi_\alpha^0(\tau_b) \subseteq \Sigma_\beta^0(\tau_b) \cap \Pi_\beta^0(\tau_b).$$

PROOF. By Lemma 2.2 it is enough to show that $\tau_b = \Sigma_1^0(\tau_b) \subseteq \Sigma_2^0(\tau_b)$. Given U open and $\nu < \kappa$, the set $D_\nu := \bigcup \{N_s \mid N_s \subseteq U \wedge \text{lh}(s) = \nu\}$ is τ_b -clopen, and since $U = \bigcup_{\nu < \kappa} D_\nu$, then $U \in \Sigma_2^0(\tau_b)$. \square

COROLLARY 4.16. *Assume AC and that $2^\kappa < 2^{(2^{<\kappa})}$. Then for $1 \leq \alpha < \kappa^+$, neither $\Sigma_\alpha^0(\tau_b)$ nor $\Pi_\alpha^0(\tau_b)$ have a universal set.*

PROOF. By Lemmas 3.10(b) and 4.15, $\Sigma_\alpha^0(\tau_b)$ has size $\geq 2^{(2^{<\kappa})}$, while $\{\mathcal{U}^{(y)} \mid y \in {}^\kappa 2\}$ has size $\leq 2^\kappa$ for all $\mathcal{U} \subseteq {}^\kappa 2 \times {}^\kappa 2$. The case of $\Pi_\alpha^0(\tau_b)$ follows by taking complements. \square

Note that Corollary 4.16 applies e.g. when $\kappa = \omega_1$ and \mathbf{MA}_{ω_1} holds.

Although there may be no universal sets for $\Sigma_\alpha^0(\tau_b)$ and $\Pi_\alpha^0(\tau_b)$, we always have \leq_L^κ -complete sets for such classes. (We use here the notation and terminology from Section 3.2.1.) For $x, y \in {}^\kappa 2$, let $x \oplus y \in {}^\kappa 2$ be defined by

$$(x \oplus y)(\alpha) := \begin{cases} x(\beta) & \text{if } \alpha = 2\beta \\ y(\beta) & \text{if } \alpha = 2\beta + 1. \end{cases}$$

For $A \subseteq {}^\kappa 2$ and $x \in {}^\kappa 2$, let

$$x \oplus A := \{x \oplus a \mid a \in A\} \subseteq {}^\kappa 2.$$

Using games, it is easy to check that $A \leq_L^\kappa x \oplus A$ and that if A has empty interior, then $x \oplus A \leq_W^\kappa A$.

LEMMA 4.17 (AC). *Consider the space $({}^\kappa 2, \tau_b)$. There is a sequence $\langle A_\alpha \mid 1 \leq \alpha < \kappa^+ \rangle$ such that each $A_\alpha \subseteq {}^\kappa 2$ is \leq_L^κ -complete for $\Sigma_\alpha^0(\tau_b)$.*

¹⁸At least assuming $\text{AC}_\omega(\mathbb{R})$ — see [Mil11].

PROOF. The set $A_1 := {}^\kappa 2 \setminus \{0^{(\kappa)}\}$ is open, and if U is open then **II** wins $G_L(U, A_1)$ by playing 0's as long as **I** has not reached a position $s \in {}^{<\kappa} 2$ such that $N_s \subseteq U$ — if this ever happens then **II** plays a 1 and then plays an arbitrary sequence.

Suppose $\alpha > 1$. Choose $\langle \alpha_\nu \mid \nu < \kappa \rangle$ such that $1 \leq \alpha_\nu < \alpha$, $\alpha = \sup_{\nu < \kappa} (\alpha_\nu + 1)$, and

$$|\{\nu < \kappa \mid \alpha_\nu = \gamma\}| \in \{0, \kappa\}$$

for all $\gamma < \alpha$. Let $p_\nu: {}^\kappa 2 \rightarrow {}^\kappa 2$ be the map $p_\nu(x)(\beta) := x(\langle \nu, \beta \rangle)$, where $\langle \cdot, \cdot \rangle$ is the pairing map of (2.1). Let

$$A_\alpha := \{x \in {}^\kappa 2 \mid \exists \nu < \kappa (p_\nu(x) \in {}^\kappa 2 \setminus A_{\alpha_\nu})\}.$$

As each p_ν is continuous, $A_\alpha \in \Sigma_\alpha^0(\tau_b)$, so it is enough to show that $B \leq_L^\kappa A_\alpha$ for all $B \in \Sigma_\alpha^0(\tau_b)$. Suppose $B = \bigcup_{\xi < \kappa} B_\xi$ with $B_\xi \in \Pi_{\beta_\xi}^0(\tau_b)$ and $\beta_\xi < \alpha$. Let $g: \kappa \rightarrow \kappa$ be such that $\beta_\xi \leq \alpha_{g(\xi)}$ — such a g exists by our choice of the α_ν 's. Construct a sequence of Lipschitz functions $\langle f_\nu \mid \nu < \kappa \rangle$ as follows:

$$(4.4a) \quad \text{if } \nu \in \text{ran } g \text{ and } g(\xi) = \nu, \text{ then } f_\nu \text{ witnesses } B_\xi \leq_L^\kappa {}^\kappa 2 \setminus A_{\alpha_\nu};$$

$$(4.4b) \quad \text{if } \nu \notin \text{ran } g, \text{ then } f_\nu \text{ is constant and takes value in } A_{\alpha_\nu}.$$

The specific bijection $\langle \cdot, \cdot \rangle$ maps $\kappa \times \kappa$ onto κ and guarantees that the function

$$f: {}^\kappa 2 \rightarrow {}^\kappa 2, \quad f(x)(\langle \nu, \beta \rangle) := (f_\nu(x))(\beta)$$

is Lipschitz and since $p_\nu(f(x))(\beta) = f(x)(\langle \nu, \beta \rangle) = f_\nu(x)(\beta)$, then

$$\forall \nu < \kappa [p_\nu(f(x)) = f_\nu(x)].$$

If $x \in B$ then $x \in B_\xi$ for some $\xi < \kappa$, so $f_\nu(x) \notin A_{\alpha_\nu}$ by (4.4a) where $\nu := g(\xi)$, and hence $f(x) \in A_\alpha$. Conversely, if $p_\nu(f(x)) \notin A_{\alpha_\nu}$ for some $\nu < \kappa$, then $\nu \in \text{ran } g$ by (4.4b), thus $\nu = g(\xi)$ for some ξ , and hence $x \in B_\xi$ by the choice of f_ν and therefore $x \in B$. This completes the proof that $\forall x \in {}^\kappa 2 [x \in B \Leftrightarrow f(x) \in A_\alpha]$. \square

Note that by Lemma 4.15, if A_α, A_β are as in Lemma 4.17 then $A_\alpha \leq_L^\kappa A_\beta$ for $1 \leq \alpha \leq \beta < \kappa^+$.

THEOREM 4.18 (AC). *For every infinite cardinal κ , $\mathbf{B}_{\kappa+1}({}^\kappa 2, \tau_b) \neq \mathcal{P}({}^\kappa 2)$.*

PROOF. Let $\langle r_\alpha \mid \alpha < \kappa^+ \rangle$ be distinct elements of ${}^\kappa 2$. Let

$$A := \bigcup_{\alpha < \kappa^+} r_\alpha \oplus A_\alpha$$

where A_α is as in Lemma 4.17. By Proposition 3.9(a) it is enough to check that A is \leq_L^κ -hard for $\mathbf{B}_{\kappa+1}({}^\kappa 2, \tau_b)$. If $B \in \mathbf{B}_{\kappa+1}({}^\kappa 2, \tau_b)$ then $B \in \Sigma_\alpha^0(\tau_b)$ for some $\alpha < \kappa^+$, and hence $B \leq_L^\kappa A_\alpha$ by Lemma 4.17. Since $A_\alpha \leq_L^\kappa A$ we are done. \square

PROPOSITION 4.19 (AC). *Consider the space $({}^\kappa 2, \tau_b)$ and let $1 \leq \alpha < \beta < \kappa^+$. Then $\Sigma_\alpha^0(\tau_b) \neq \Pi_\alpha^0(\tau_b)$ and $\Sigma_\alpha^0(\tau_b) \subset \Sigma_\beta^0(\tau_b)$.*

PROOF. For the first part, argue by contradiction and use Lemma 4.17 and Proposition 3.9(b). The second part follows from the first one and Lemma 4.15. \square

This shows that, under AC, the $\kappa+1$ -Borel hierarchy on $({}^\kappa 2, \tau_b)$ never collapses, independently of the choice of the cardinal κ . However, note that $\Sigma_1^0(\tau_b) \neq \Pi_1^0(\tau_b)$ already holds in ZF since e.g. ${}^\kappa 2 \setminus \{0^{(\kappa)}\}$ is open but not closed.

As for the classical case $\kappa = \omega$, using the fact that the classes $\Sigma_\alpha^0(\tau_b)$ and $\Pi_\alpha^0(\tau_b)$ are closed under continuous preimages, we get the following corollary, which is a strengthening of Proposition 3.9(b) for the special case $\mathcal{S} := \mathbf{B}_{\kappa+1}({}^\kappa 2, \tau_b)$.

COROLLARY 4.20. *There is no \leq_W^κ -complete set for $\mathbf{B}_{\kappa+1}(\kappa 2, \tau_b)$, i.e. there is no $A \in \mathbf{B}_{\kappa+1}(\kappa 2, \tau_b)$ such that $B \leq_W^\kappa A$ for every $B \in \mathbf{B}_{\kappa+1}(\kappa 2, \tau_b)$.*

4.3.2. *Cardinality of $\mathbf{B}_{\kappa+1}$.* Cardinality considerations may be useful to show that the notion of $\kappa + 1$ -Borelness is nontrivial.

PROPOSITION 4.21 (AC). *Let κ be an infinite cardinal.*

- (a) $|\mathbf{B}_{\kappa+1}(\omega 2)| = \min\{2^\kappa, 2^{(2^{\aleph_0})}\}$. Therefore $2^\kappa < 2^{(2^{\aleph_0})} \Leftrightarrow |\mathbf{B}_{\kappa+1}(\omega 2)| < |\mathcal{P}(\omega 2)|$.
- (b) $|\mathbf{B}_{\kappa+1}(\kappa 2, \tau_p)| = 2^\kappa = |\tau_p|$. Therefore $|\mathbf{B}_{\kappa+1}(\kappa 2, \tau_p)| < 2^{(2^\kappa)}$, and hence, in particular, $\mathbf{B}_{\kappa+1}(\kappa 2, \tau_p) \neq \mathcal{P}(\kappa 2)$.
- (c) $|\mathbf{B}_{\kappa+1}(\kappa 2, \tau_b)| = 2^{(2^{<\kappa})} = |\tau_b|$. Therefore $2^{(2^{<\kappa})} < 2^{(2^\kappa)} \Leftrightarrow |\mathbf{B}_{\kappa+1}(\kappa 2, \tau_b)| < |\mathcal{P}(\kappa 2)|$.

PROOF. (a) $|\mathbf{B}_{\kappa+1}(\omega 2)| \leq (2^{\aleph_0})^\kappa = 2^\kappa$ by Lemma 2.3(b), and since $\mathbf{B}_{\kappa+1}(\omega 2) \subseteq \mathcal{P}(\omega 2)$ then

$$|\mathbf{B}_{\kappa+1}(\omega 2)| \leq \min\{2^\kappa, 2^{(2^{\aleph_0})}\}.$$

To prove the reverse inequality we take cases. If $\kappa \geq 2^{\aleph_0}$ then $\mathbf{B}_{\kappa+1}(\omega 2) = \mathcal{P}(\omega 2)$, and we are done. If $\kappa < 2^{\aleph_0}$, let $\langle x_\alpha \mid \alpha < \kappa \rangle$ be distinct elements of ${}^\omega 2$: then the map $\mathcal{P}(\kappa) \rightarrow \mathbf{B}_{\kappa+1}(\omega 2)$, $A \mapsto \{x_\alpha \mid \alpha \in A\}$ is injective.

(b) By Lemmas 3.10(a) and 2.3, we get $2^\kappa = |\tau_p| \leq |\mathbf{B}_{\kappa+1}(\kappa 2, \tau_p)| \leq (2^\kappa)^\kappa = 2^\kappa$.

(c) By Lemmas 3.10(b) and 2.3, and by $\kappa \leq 2^{<\kappa}$, we get $2^{(2^{<\kappa})} = |\tau_b| \leq |\mathbf{B}_{\kappa+1}(\kappa 2, \tau_b)| \leq (2^{(2^{<\kappa})})^\kappa = 2^{(2^{<\kappa})}$. \square

REMARK 4.22. By parts (b) and (c) of Proposition 4.21, if $2^\kappa < 2^{(2^{<\kappa})}$ (which follows from MA_{ω_1} when $\kappa = \omega_1$) then $\mathbf{B}_{\kappa+1}(\kappa 2, \tau_p) \neq \mathbf{B}_{\kappa+1}(\kappa 2, \tau_b)$. This should be contrasted with Corollary 4.10.

5. Generalized Borel functions

5.1. Basic facts.

5.1.1. *Borel measurable functions.* We now turn to α -Borel measurable functions between generalized Cantor spaces. Unless otherwise explicitly stated (see e.g. the definition of a weakly \mathcal{S} -measurable function below), in this section such spaces are tacitly endowed with the bounded topology τ_b , so that all other derived topological notions (such as α -Borelness, and so on) refer to such topology.

DEFINITION 5.1. Let λ, μ be infinite cardinals, let $f: {}^\lambda 2 \rightarrow {}^\mu 2$, and let $\mathcal{S} \subseteq \mathcal{P}({}^\lambda 2)$ be an algebra.

- f is **\mathcal{S} -measurable** if and only if the preimage of every τ_b -open subset of ${}^\mu 2$ is in \mathcal{S} .
- f is **weakly \mathcal{S} -measurable** if and only if the preimage of every τ_p -open subset of ${}^\mu 2$ is in \mathcal{S} .
- If α is an infinite ordinal and $\mathcal{S} = \mathbf{B}_\alpha({}^\lambda 2)$, a (weakly) \mathcal{S} -measurable function is called a **(weakly) α -Borel function**. Therefore f is an α -Borel function if and only if $f^{-1}(B) \in \mathbf{B}_\alpha({}^\lambda 2)$ for every $B \in \mathbf{B}_\alpha({}^\mu 2)$.

If in Definition 5.1 α -Borel sets are replaced with their effective versions, we get the notion of **effective (weakly) α -Borelness** for functions $f: {}^\lambda 2 \rightarrow {}^\mu 2$. As for the case of sets, this notion coincides with its non-effective version under AC, but it may be a stronger notion in a choice-less world. Every (effective) α -Borel function is (effective) weakly α -Borel because τ_b refines τ_p . On the other hand, if $f: {}^\lambda 2 \rightarrow {}^\mu 2$ is weakly α -Borel, then f might fail to be α -Borel (see Proposition 5.2(b) below), but if $\alpha \geq \mu$ then at least all preimages of τ_b -basic open sets of ${}^\mu 2$ are in $\mathbf{B}_\alpha({}^\lambda 2)$, that is

$$\forall U \in \mathcal{B}_b({}^\mu 2) \ (f^{-1}(U) \in \mathbf{B}_\alpha({}^\lambda 2)).$$

(Use the fact that every τ_b -basic open set of ${}^\mu 2$ can be written as an intersection of $< \mu$ -many τ_p -clopen sets). This also implies that when $\alpha > \mu$, then the preimage under f of any set in the $\mu + 1$ -algebra¹⁹ $\mathbf{B}_{\mu+1}(\mathcal{B}_b)$ on ${}^\mu 2$ generated by the canonical basis $\mathcal{B}_b({}^\mu 2)$ belongs to $\mathbf{B}_\alpha({}^\lambda 2)$.

Proposition 5.2(a) shows that under AC the notions α -Borel and weakly α -Borel may also coincide if certain cardinal conditions are satisfied.

PROPOSITION 5.2. *Let λ, μ be cardinals, $f: {}^\lambda 2 \rightarrow {}^\mu 2$, and $\alpha \geq \omega$.*

- (a) *Assume AC. If $2^{<\mu} = \mu < \alpha$ and f is (effective) weakly α -Borel, then f is (effective) α -Borel. In particular, if $\mu = \omega$ (here we do not need AC), or if CH holds and $\mu = \omega_1$, then f is (effective) weakly α -Borel if and only if f is (effective) α -Borel.*
- (b) *If $\lambda < \mu$ and $\mathbf{B}_\alpha({}^\lambda 2) \neq \mathcal{P}({}^\lambda 2)$, then the inclusion map $i: {}^\lambda 2 \hookrightarrow {}^\mu 2$ is weakly α -Borel but not α -Borel.*

PROOF. Part (a) easily follows from the fact that $\mathcal{B}_b({}^\mu 2)$ has size $2^{<\mu}$.

(b) The inclusion function is not α -Borel by an argument as in Proposition 3.14. By Lemma 3.15(a) we get that $i: ({}^\lambda 2, \tau_b) \hookrightarrow ({}^\mu 2, \tau_p)$ is continuous, and hence i is trivially weakly α -Borel. \square

Since the inclusion map of part (b) of Proposition 5.2 is arguably one of the simplest functions that one may want to consider, this suggests that weakly α -Borelness is a more natural and appropriate notion of topological complexity for functions ${}^\lambda 2 \rightarrow {}^\mu 2$ when $\lambda < \mu$. On the other hand, it seems that in general there is no obstruction to the possibility of considering the stronger notion of α -Borelness when $\lambda \geq \mu$ (we come back to this point at the beginning of Section 14.1).

As the name suggests, the notion of weakly α -Borelness is quite weak. In fact there are situations where it becomes vacuous, i.e. every function is weakly α -Borel. Proposition 5.5 shows that the existence of a non-weakly α -Borel $f: {}^\lambda 2 \rightarrow {}^\mu 2$ depends only on λ and α .

5.1.2. **Γ -in-the-codes functions.** When considering projective levels in models of AD, it is natural to code functions of the form $f: {}^\omega 2 \rightarrow {}^\kappa 2$ (for suitable $\kappa < \Theta$) as subsets of (products of) ${}^\omega 2$ and ${}^\omega \omega$, and then require that such a code be in the projective pointclass under consideration. More precisely, we have the following general definition, which makes sense in arbitrary models of ZF. To simplify the presentation, we say that ρ is a **Γ -code for κ** if ρ is a **Γ -norm** of length κ on some $A \in \Gamma({}^\omega \omega)$.

DEFINITION 5.3. Let Γ be a boldface pointclass, let κ be an infinite cardinal, and assume that there is a Γ -code ρ for κ . We say that the function $f: {}^\omega 2 \rightarrow {}^\kappa 2$ is **Γ -in-the-codes** (with respect to ρ) if there is $F \in \Gamma({}^\omega 2 \times {}^\omega \omega \times 2)$ such that for every $\alpha < \kappa$ and $i \in 2$

$$f^{-1}(\tilde{\mathbf{N}}_{\alpha,i}^\kappa) = \{x \in {}^\omega 2 \mid \exists y \in A (\rho(y) = \alpha \wedge (x, y, i) \in F)\},$$

where $\tilde{\mathbf{N}}_{\alpha,i}^\kappa$ is defined as in Remark 3.2(ii).

REMARKS 5.4. (i) As we shall see in Remark 5.11(ii), Definition 5.3 does not really depend on the choice of ρ when Γ is sufficiently closed.

- (ii) If there is a Γ -code ρ for κ , then $\kappa \leq \delta_\Gamma$ (see Definition 4.11). Indeed, if $\alpha < \kappa$ and $z \in A$ is such that $\rho(z) = \alpha$, then the prewellordering

$$x \preceq y \Leftrightarrow [y \in A \wedge \rho(y) < \rho(z) \Rightarrow x \in A \wedge \rho(x) \leq \rho(y)]$$

¹⁹In [FHK14, Definition 15] the authors call $(\mu + 1)$ -Borel sets the elements of $\mathbf{B}_{\mu+1}(\mathcal{B}_b)$, rather than those in $\mathbf{B}_{\mu+1}(\tau_b)$, thus the functions $f: {}^\mu 2 \rightarrow {}^\mu 2$ which are weakly $\mu + 1$ -measurable in our sense are still (generalized) Borel functions in their sense. Note that Definition 15 of [FHK14] coincides with our Definition 4.1 when $\mu^{<\mu} = \mu$, which is the setup of that paper.

is in Δ_{Γ} and has length $\alpha + 1$. On the other hand, if $\kappa < \delta_{\Gamma}$ then there is a Γ -code ρ for κ : it is enough to let ρ be the Γ -norm associated to any Δ_{Γ} prewellordering of ${}^{\omega}\omega$ of length κ , which exists by $\kappa < \delta_{\Gamma}$.

- (iii) Let $\Gamma \subseteq \Gamma'$ be boldface pointclasses. If there is a Γ -code for κ , then there is also a Γ' -code for any $\kappa' \leq \kappa$.

If $\Gamma = \Sigma_1^1$ in Definition 5.3, then $\kappa = \delta_1^1 = \omega$ by Remark 5.4(ii) and [Mos09, Theorem 4A.4], and therefore a function $f: {}^{\omega}2 \rightarrow {}^{\omega}2$ is Σ_1^1 -in-the-codes if and only if it is $(\omega+1)$ -Borel measurable. More generally, if Γ is a nontrivial boldface pointclass closed under projections and countable intersections (e.g. if $\Gamma = \Sigma_n^1$ for some $n \geq 1$), then for all $f: {}^{\omega}2 \rightarrow {}^{\omega}2$

$$f \text{ is } \Gamma\text{-in-the-codes} \Leftrightarrow f \text{ is } \Gamma\text{-measurable} \Leftrightarrow f \text{ is } \Delta_{\Gamma}\text{-measurable.}$$

To see this, use the fact that under the hypotheses above we have that f is Γ -measurable if and only if the graph of f is in $\Gamma({}^{\omega}2 \times {}^{\omega}2)$. This result can be partially generalized to uncountable κ 's. In fact, since ${}^{\omega}2 \setminus f^{-1}(\tilde{N}_{\alpha,i}^{\kappa}) = f^{-1}(\tilde{N}_{\alpha,1-i}^{\kappa})$, if Γ is closed under projections and finite intersections then

$$(5.1) \quad f: {}^{\omega}2 \rightarrow {}^{\kappa}2 \text{ is } \Gamma\text{-in-the-codes} \Rightarrow \forall \alpha < \kappa \forall i \in \{0, 1\} (f^{-1}(\tilde{N}_{\alpha,i}^{\kappa}) \in \Delta_{\Gamma}).$$

Lemma 5.8 and Proposition 5.10 below show that for certain Γ 's, the implication can be reversed.

5.2. * Further results.

PROPOSITION 5.5. *For α an infinite ordinal, the following are equivalent:*

- (a) $\mathbf{B}_{\alpha}({}^{\lambda}2) \neq \mathcal{P}({}^{\lambda}2)$,
- (b) for all cardinals μ there is an $f: {}^{\lambda}2 \rightarrow {}^{\mu}2$ which is not weakly α -Borel,
- (c) there is a cardinal μ and there is an $f: {}^{\lambda}2 \rightarrow {}^{\mu}2$ which is not weakly α -Borel.

PROOF. If $A \subseteq {}^{\lambda}2$ is not in $\mathbf{B}_{\alpha}({}^{\lambda}2)$, then picking distinct $y_0, y_1 \in {}^{\mu}2$ one easily sees that the function mapping the points in A to y_1 and the points in ${}^{\lambda}2 \setminus A$ to y_0 is not weakly α -Borel. This proves that (a) \Rightarrow (b).

Since (b) \Rightarrow (c) is trivial, it is enough to prove that (a) follows from (c): but this is easy, as if $f: {}^{\lambda}2 \rightarrow {}^{\mu}2$ is not weakly α -Borel then by definition $f^{-1}(U) \notin \mathbf{B}_{\alpha}({}^{\lambda}2)$ for some τ_p -open $U \subseteq {}^{\mu}2$. \square

We are mainly interested in (weakly) $\kappa + 1$ -Borel functions between the Cantor space ${}^{\omega}2$ and (a homeomorphic copy of) ${}^{\kappa}2$, with $\kappa > \omega$ a cardinal. From Propositions 4.21 and 5.5 we obtain

COROLLARY 5.6. *Assume AC.*

- (a) If $\kappa \geq 2^{\aleph_0}$, then every $f: {}^{\omega}2 \rightarrow {}^{\kappa}2$ is $\kappa + 1$ -Borel.
- (b) If $2^{\kappa} < 2^{(2^{\aleph_0})}$, then there is an $f: {}^{\omega}2 \rightarrow {}^{\kappa}2$ which is not weakly $\kappa + 1$ -Borel.
- (c) If $2^{(2^{<\kappa})} < 2^{(2^{\kappa})}$, then there is an $f: {}^{\kappa}2 \rightarrow {}^{\omega}2$ which is not weakly $\kappa + 1$ -Borel.

Since $\kappa + 1$ -Borel functions are in particular weakly $\kappa + 1$ -Borel, we could have removed the adjective “weakly” in parts (b) and (c) of Corollary 5.6.

From Propositions 4.6 and 5.5 and equation (4.2) on page 31 we obtain

COROLLARY 5.7. *Assume AD.*

- (a) If $\kappa < \Theta$, then there is an $f: {}^{\omega}2 \rightarrow {}^{\kappa}2$ which is not weakly $\kappa + 1$ -Borel.
- (b) If $\kappa = \lambda_{2n+1}^1$ as defined in condition (C) on page 31, then $f: {}^{\omega}2 \rightarrow {}^{\kappa}2$ is (weakly) $\kappa + 1$ -Borel if and only if it is (weakly) Δ_{2n+1}^1 -measurable.

We now turn our attention to Γ -in-the-codes functions.

LEMMA 5.8. *Let Γ be a nontrivial boldface pointclass closed under projections and finite intersections and unions. Let κ be an infinite cardinal, and suppose that Γ is closed under well-ordered unions of length κ and that there is a Γ -code ρ for κ . Then for every $f: {}^\omega 2 \rightarrow {}^\kappa 2$ the following are equivalent:*

- (a) *f is Γ -in-the-codes (with respect to ρ);*
- (b) *$f^{-1}(\tilde{N}_{\alpha,i}^\kappa) \in \Delta_\Gamma$ for every $\alpha < \kappa$ and $i = 0, 1$;*
- (c) *$f^{-1}(U) \in \Delta_\Gamma$ for every $U \in \mathcal{B}_p({}^\kappa 2)$.*

PROOF. The equivalence between (b) and (c) is easy, while (b) follows from (a) by equation (5.1). So it is enough to show that (b) implies (a). If ρ is as in the hypotheses and $A \in \Gamma({}^\omega \omega)$ is its domain, it is enough to prove that the set $\bigcup_{\alpha < \kappa, i \in 2} f^{-1}(\tilde{N}_{\alpha,i}^\kappa) \times \{y \in A \mid \rho(y) = \alpha\} \times \{i\}$ is in Γ . But this easily follows from the fact that by (b) each $f^{-1}(\tilde{N}_{\alpha,i}^\kappa) \times \{y \in A \mid \rho(y) = \alpha\} \times \{i\}$ belongs to $\Delta_\Gamma \subseteq \Gamma$, together with the fact that Γ is closed under well-ordered unions of length κ . \square

LEMMA 5.9 (AD + DC). *Let Γ be a nontrivial boldface pointclass closed under projections, countable unions, and countable intersections. Let κ be an infinite cardinal, and suppose that there is a Γ -code for κ . Then Γ is closed under well-ordered unions of length κ .*

PROOF. Let ρ be a Γ -code for κ and $A \in \Gamma({}^\omega \omega)$ be its domain. Recall from Remark 5.4(ii) that the existence of such a ρ implies $\kappa \leq \delta_\Gamma$. If $\kappa < \delta_\Gamma$ we can apply Moschovakis' Coding Lemma (see [Mos09, Lemmas 7D.5 and 7D.6]). If $\kappa = \delta_\Gamma$, then $A \notin \tilde{\Gamma}({}^\omega \omega)$ by Fact 4.12 and by Wadge's lemma every $B \in \Gamma({}^\omega \omega)$ is of the form $g^{-1}(A)$ for some continuous function $g: {}^\omega \omega \rightarrow {}^\omega \omega$, and hence $\rho \circ g$ is a Γ -norm on B . Therefore the pointclass Γ has the prewellordering property, and since Γ is assumed to be closed under projections and A witnesses $\Gamma({}^\omega \omega) \neq \tilde{\Gamma}({}^\omega \omega)$, this implies that Γ is closed under well-ordered unions of arbitrary length by e.g. [Jac08, Theorem 2.16]. \square

Thus in the AD world Lemma 5.8 can be reformulated as follows.

PROPOSITION 5.10 (AD + DC). *Let Γ be a nontrivial boldface pointclass closed under projections, countable unions, and countable intersections. Let κ be an infinite cardinal, and suppose that there is a Γ -code ρ for κ . Then for every $f: {}^\omega 2 \rightarrow {}^\kappa 2$ the following are equivalent:*

- (a) *f is Γ -in-the-codes (with respect to ρ);*
- (b) *$f^{-1}(\tilde{N}_{\alpha,i}^\kappa) \in \Delta_\Gamma$ for every $\alpha < \kappa$ and $i = 0, 1$;*
- (c) *$f^{-1}(U) \in \Delta_\Gamma$ for every $U \in \mathcal{B}_p({}^\kappa 2)$.*

If Γ is a nontrivial boldface pointclass closed under projections, countable unions, and countable intersections, Proposition 5.10 can be applied when

- $\kappa < \delta_\Gamma$,
- $\kappa = \delta_\Gamma$ and Γ is a Spector pointclass closed under co-projections,
- $\Gamma = \Sigma_n^1$ and $\kappa^+ = \delta_n^1$ (i.e. $\kappa = \lambda_n^1$ if n is odd and $\kappa = \delta_{n-1}^1$ if $n > 0$ is even),
- $\Gamma = \Sigma_1^2$ and $\kappa = \delta_1^2$.

As we shall see in Proposition 9.10 and Corollary 9.29, in all these (and other) interesting cases the Γ -in-the-codes functions $f: {}^\omega 2 \rightarrow {}^\kappa 2$ turn out to be just a special case of weakly $\kappa + 1$ -Borel functions, and for the odd levels of the projective hierarchy we further have that in fact the two notions coincide.

REMARKS 5.11. (i) If $\Gamma = \Sigma_1^1$ then AD is not needed in Proposition 5.10 and the proof goes through in $\text{ZF} + \text{AC}_\omega(\mathbb{R})$.

- (ii) Definition 5.3 seems to depend on the particular choice of the Γ -norm ρ : however, since Lemma 5.8(b) and Proposition 5.10(b) above do not depend on ρ , Lemma 5.8 and Proposition 5.10 show that this is not the case for any sufficiently closed nontrivial boldface pointclass Γ .

6. The generalized Baire space and Baire category

6.1. The generalized Baire space. So far we just considered the generalized Cantor space ${}^\kappa 2$, but similar results hold for the generalized Baire space ${}^\kappa \kappa$, with κ an uncountable cardinal.

DEFINITION 6.1. Let $\widehat{N}_s^\kappa := \{x \in {}^\kappa \kappa \mid s \subseteq x\}$, where $s: u \rightarrow \kappa$ and $u \subseteq \kappa$.

- The **bounded topology** τ_b on ${}^\kappa \kappa$ is the one generated by the collection

$$\widehat{B}_b := \{\widehat{N}_s^\kappa \mid s \in {}^{<\kappa} \kappa\}.$$

The sets \widehat{N}_s^κ with $s: u \rightarrow \kappa$ and u a *bounded* subset of κ , form a basis for τ_b as well.

- The **product topology** τ_p on ${}^\kappa \kappa$ is the product of κ copies of the space κ with the discrete topology. A basis for τ_p is

$$\widehat{B}_p := \{\widehat{N}_s^\kappa \mid s: u \rightarrow \kappa \wedge u \in [\kappa]^{<\omega}\}.$$

- For $\omega \leq \lambda \leq \kappa$, the **λ -topology** τ_λ on ${}^\kappa \kappa$ is the one generated by the collection

$$\widehat{B}_\lambda := \{\widehat{N}_s^\kappa \mid s: u \rightarrow \kappa \wedge u \in [\kappa]^{<\lambda}\},$$

with the proviso as in Convention 3.5 that

$$(6.1) \quad \kappa = \text{cof}(\kappa) \Rightarrow \widehat{B}_\kappa := \{\widehat{N}_s^\kappa \mid s \in {}^{<\kappa} \kappa\} = \widehat{B}_b.$$

If X is a subspace of $({}^\kappa \kappa, \tau_*)$ where $*$ is one of b, p , or λ , then the relative topology on X is still denoted by τ_* , so that when $X = {}^\kappa 2$ this agrees with Definitions 3.1 and 3.3. Another subspace of ${}^\kappa \kappa$ of interest to us is

$$\text{Sym}(\kappa),$$

the group of all permutations of κ , which turns out to be an intersection of κ -many τ_p -open (and hence also τ_λ -open and τ_b -open) sets, and hence a $\Pi_2^0(\tau_*)$ set, where $*$ $\in \{b, p, \lambda\}$.

Most of the observations and results on ${}^\kappa 2$ seen in the previous sections hold *mutatis mutandis* for ${}^\kappa \kappa$. We summarize some of them in the following proposition, in which all the topologies are understood to be on ${}^\kappa \kappa$.

PROPOSITION 6.2. *Let κ be an uncountable cardinal, and let $\omega \leq \lambda < \max(\text{cof}(\kappa)^+, \kappa)$.*

- If $\lambda < \nu < \max(\text{cof}(\kappa)^+, \kappa)$ then $\tau_p = \tau_\omega \subseteq \tau_\lambda \subset \tau_\nu \subseteq \tau_{\text{cof}(\kappa)} \subseteq \tau_b$, and $\tau_b = \tau_{\text{cof}(\kappa)}$ if and only if κ is regular.*
- $|\tau_p| = |\mathcal{P}(\kappa)|$, and $|\tau_b| = |\mathcal{P}({}^{<\kappa} \kappa)|$. Therefore,*
 - assuming AD, $\kappa < \Theta \Rightarrow |\tau_p| < |\tau_b|$,*
 - assuming AC, $|\tau_p| < |\tau_b| \Leftrightarrow 2^\kappa < 2^{(\kappa^{<\kappa})}$.*
- The topologies $\tau_p, \tau_\lambda, \tau_b$ are perfect, regular Hausdorff, and zero-dimensional, and they are never κ -compact.*
- The topology τ_* is not first countable (and hence neither second countable nor metrizable) and also not separable, for $*$ $\in \{p, \lambda\}$.*
- The topology τ_b is neither second countable nor separable, and in fact assuming AC it has density $\kappa^{<\kappa}$. It is first countable if and only if it is metrizable if and only if it is completely metrizable if and only if $\text{cof}(\kappa) = \omega$.*

- (f) The topology τ_* with $*$ $\in \{p, \lambda, b\}$ is closed under intersections of length $\leq \alpha$ (for some ordinal α) if and only if: $\alpha < \omega$ when $*$ $= p$, $\alpha < \lambda$ when $*$ $= \lambda$ (assuming AC), $|\alpha| < \text{cof}(\kappa)$ when $*$ $= b$. Therefore the collection of all τ_* -clopen subsets of ${}^\kappa\kappa$ is a ω -algebra if $*$ $= p$, is a λ -algebra (assuming AC) if $*$ $= \lambda$, and is a $\text{cof}(\kappa)$ -algebra if $*$ $= b$.
- (g) Assume AC. Then $\mathbf{B}_{\kappa+1}({}^\kappa\kappa, \tau_b) \neq \mathcal{P}({}^\kappa\kappa)$. Moreover $\Sigma_\alpha^0({}^\kappa\kappa, \tau_b) \neq \Pi_\alpha^0({}^\kappa\kappa, \tau_b)$ and $\Sigma_\alpha^0({}^\kappa\kappa, \tau_b) \subset \Sigma_\beta^0({}^\kappa\kappa, \tau_b)$, for $1 \leq \alpha < \beta < \kappa^+$.
- (h) Assume AC. Then $|\mathbf{B}_{\kappa+1}({}^\kappa\kappa, \tau_p)| = 2^\kappa = |\tau_p|$ and $|\mathbf{B}_{\kappa+1}({}^\kappa\kappa, \tau_b)| = 2^{(\kappa^{<\kappa})} = |\tau_b|$. Therefore, $|\mathbf{B}_{\kappa+1}({}^\kappa\kappa, \tau_p)| < 2^{(2^\kappa)}$ and $|\mathbf{B}_{\kappa+1}({}^\kappa\kappa, \tau_b)| < |\mathcal{P}({}^\kappa\kappa)| \Leftrightarrow 2^{(\kappa^{<\kappa})} < 2^{(2^\kappa)}$. In particular, $\mathbf{B}_{\kappa+1}({}^\kappa\kappa, \tau_p) \neq \mathcal{P}({}^\kappa\kappa)$ and $2^{(\kappa^{<\kappa})} < 2^{(2^\kappa)} \Rightarrow \mathbf{B}_{\kappa+1}({}^\kappa\kappa, \tau_b) \neq \mathcal{P}({}^\kappa\kappa)$.
- (i) A subset of ${}^\kappa\kappa$ is (effective) ${}^{<\kappa}\kappa + 1$ -Borel with respect to τ_p if and only if it is (effective) ${}^{<\kappa}\kappa + 1$ -Borel with respect to τ_b . Therefore, if we assume AC and $\kappa^{<\kappa} = \kappa$ then $\mathbf{B}_{\kappa+1}({}^\kappa\kappa, \tau_p) = \mathbf{B}_{\kappa+1}({}^\kappa\kappa, \tau_b)$.

REMARKS 6.3. (i) Part (d) of Proposition 6.2 should be contrasted with part (e) of Proposition 3.12: this is one of the few differences between the generalized Cantor and Baire spaces.

- (ii) Part (g) of Proposition 6.2 follows from Theorem 4.18 and Proposition 4.19 together with the fact that ${}^\kappa 2$ is a closed subspace of ${}^\kappa\kappa$ and we are dealing with hereditary general boldface pointclasses.
- (iii) If $\kappa < \kappa^{<\kappa}$, then it may be the case that $|\mathbf{B}_{\kappa+1}({}^\kappa\kappa, \tau_p)| < |\mathbf{B}_{\kappa+1}({}^\kappa\kappa, \tau_b)|$ (see Remark 4.22).
- (iv) The proof that $({}^\kappa\kappa, \tau_b)$ is (completely) metrizable if and only if $\text{cof}(\kappa) = \omega$ (part (e) of Proposition 6.2) follows the same ideas as the proof of part (h) of Proposition 3.12: it is enough to identify ${}^\kappa\kappa$ with ${}^\kappa G$ with G a group of size κ and then use the Birkhoff-Kakutani theorem, or replace $\prod_{n \in \omega} {}^{\lambda_n} 2$ with $\prod_{n \in \omega} {}^{\lambda_n} \kappa$ in the direct proof sketched in Proposition 3.12(h).

The spaces ${}^\omega 2$ and ${}^\omega \omega$ are not homeomorphic because ${}^\omega 2$ is compact while ${}^\omega \omega$ not. For the same reason, we have that also $({}^\kappa 2, \tau_p)$ and $({}^\kappa\kappa, \tau_p)$ are never homeomorphic by Proposition 3.12(b) and Proposition 6.2(c). However, we are now going to show that in models of choice the situation becomes rather different when we endow the generalized spaces ${}^\kappa 2$ and ${}^\kappa\kappa$ with the bounded topology. Recall that a regular cardinal κ is **weakly compact** if and only if it is *strong limit* (i.e. such that $2^\lambda < \kappa$ for every $\lambda < \kappa$), and has the *tree property*, that is: for every $T \subseteq {}^{<\kappa}\kappa$, if $0 < |T \cap {}^\alpha\kappa| < \kappa$ for all $\alpha < \kappa$ then there is a κ -branch through T , that is a point $x \in {}^\kappa\kappa$ such that $x \restriction \alpha \in T$ for all $\alpha < \kappa$.

LEMMA 6.4. Suppose κ is an infinite cardinal with the tree property and such that for every cardinal $\lambda < \kappa$ there is no injection from κ into ${}^\lambda 2$.

- (a) Every well-orderable τ_b -closed $C \subseteq {}^\kappa 2$ of size $\geq \kappa$ has an accumulation point in C .
- (b) There is no continuous bijection between $({}^\kappa 2, \tau_b)$ and $({}^\kappa\kappa, \tau_b)$.

PROOF. (a) Suppose towards a contradiction that all points of C are isolated in it. For each $x \in C$, let $s_x \in {}^{<\kappa} 2$ be the shortest sequence with $C \cap N_{s_x} = \{x\}$ (so that, in particular, s_x and $s_{x'}$ are incomparable whenever $x \neq x'$), and let T be the tree generated by these s_x 's, i.e.

$$T := \{t \in {}^{<\kappa} 2 \mid t \subseteq s_x \text{ for some } x \in C\}.$$

Fix $\alpha < \kappa$ and let $\lambda := |\alpha| < \kappa$. Since $\{s_x \mid x \in C\}$ is well-orderable, then $T \cap {}^\alpha 2$ is well-orderable too, and hence $|T \cap {}^\alpha 2| = \mu$ for some cardinal μ . Since ${}^\alpha 2$ and ${}^\lambda 2$ are in bijection and there is no injection from κ into ${}^\lambda 2$, we have that $\mu < \kappa$. Using a similar argument, we also get $T \cap {}^\alpha 2 \neq \emptyset$ (otherwise $s_x \in {}^{<\alpha} 2$ for each $x \in C$ and we would have an injection from $\kappa \leq |C|$ into ${}^\lambda 2$). Therefore by the tree property T has a κ -branch $x \in {}^\kappa 2$. It follows from the choice of the s_x 's

that x is an accumulation point of C , and hence also $x \in C$ because C is closed, a contradiction with our assumption that C consists only of isolated points.

(b) Let now $C' := \{\langle \alpha \rangle \wedge 0^{(\kappa)} \mid \alpha < \kappa\} \subseteq {}^\kappa \kappa$, and suppose towards a contradiction that there is a continuous bijection $f: ({}^\kappa 2, \tau_b) \rightarrow ({}^\kappa \kappa, \tau_b)$. Since C' is a closed well-orderable set of size $\geq \kappa$ with no accumulation point, then so would be $C := f^{-1}(C') \subseteq {}^\kappa 2$: but such a C cannot exist by part (a), and we are done. \square

REMARK 6.5. Notice that sets $C \subseteq {}^\kappa 2$ as in part (a) of Lemma 6.4 do exist: the set

$$C := \{0^{(\alpha)} \wedge 1 \wedge 0^{(\kappa)} \mid \alpha < \kappa\} \cup \{0^{(\kappa)}\}$$

is closed, well-orderable, and of size κ . Moreover, in Lemma 6.4(a) we cannot drop the assumption that C be closed: the set $\{0^{(\alpha)} \wedge 1 \wedge 0^{(\kappa)} \mid \alpha < \kappa\}$ is well-orderable and of size κ , but has no accumulation point in itself.

PROPOSITION 6.6 (AC). *Let κ be an infinite cardinal.*

- (a) *If κ is regular, then $({}^\kappa 2, \tau_b)$ and $({}^\kappa \kappa, \tau_b)$ are homeomorphic if and only if κ is not weakly compact (equivalently, by Remark 3.13(i), ${}^\kappa 2$ is not κ -compact).*
- (b) *If κ is singular, then $({}^\kappa 2, \tau_b)$ and $({}^\kappa \kappa, \tau_b)$ are homeomorphic if and only if κ is not strong limit (equivalently,²⁰ $2^{<\kappa} > \kappa$).*

Part (a) follows from the results in [HN73]. However, for the reader's convenience we give here a simple direct proof.

PROOF. (a) By Lemma 6.4(b), $({}^\kappa 2, \tau_b)$ and $({}^\kappa \kappa, \tau_b)$ cannot be homeomorphic if κ is weakly compact. (Alternatively, we could use [MR13, Theorem 5.6] and the fact that ${}^\kappa \kappa$ is never κ -compact by Proposition 6.2(c).) Therefore we have just to show that there is a homeomorphism $f: ({}^\kappa \kappa, \tau_b) \rightarrow ({}^\kappa 2, \tau_b)$ whenever κ is not weakly compact.

Let us first assume that there is $\lambda < \kappa$ with $2^\lambda \geq \kappa$, so that $\kappa^\lambda = 2^\lambda$, and fix a bijection $g: {}^\lambda \kappa \rightarrow {}^\lambda 2$. Given $x \in {}^\kappa \kappa$, let $\langle s_\alpha^x \mid \alpha < \kappa \rangle$ be the unique sequence of elements of ${}^\lambda \kappa$ such that $x = s_0^x \wedge s_1^x \wedge \dots \wedge s_\alpha^x \wedge \dots$. The map

$$f: {}^\kappa \kappa \rightarrow {}^\kappa 2, \quad x \mapsto g(s_0^x) \wedge g(s_1^x) \wedge \dots \wedge g(s_\alpha^x) \wedge \dots$$

is a well-defined bijection. Moreover, since the families

$$\{\widehat{N}_s \mid s \in {}^{<\kappa} \kappa, \text{lh } s = \lambda \cdot \alpha \text{ for some } \alpha < \kappa\} \subseteq \widehat{\mathcal{B}}_b({}^\kappa \kappa)$$

and

$$\{N_s \mid s \in {}^{<\kappa} 2, \text{lh } s = \lambda \cdot \alpha \text{ for some } \alpha < \kappa\} \subseteq \mathcal{B}_b({}^\kappa 2)$$

are bases for the bounded topologies on, respectively, ${}^\kappa \kappa$ and ${}^\kappa 2$, we get that $f: ({}^\kappa \kappa, \tau_b) \rightarrow ({}^\kappa 2, \tau_b)$ is a homeomorphism.

Let us now assume that κ is strong limit, so that $2^{<\kappa} = \kappa$. As κ is not weakly compact, there is a tree $T \subseteq {}^\kappa 2$ of height κ without a κ -branch. Let

$$\partial T := \{s \in {}^{<\kappa} 2 \setminus T \mid s \restriction \alpha \in T \text{ for all } \alpha < \text{lh } s\}$$

be the **boundary of T** . The sequences in ∂T are pairwise incomparable, and as T has no κ -branches, $\{N_s \mid s \in \partial T\}$ is a partition of ${}^\kappa 2$. We claim that such partition has size κ : towards a contradiction if $|\partial T| < \kappa$, then $\partial T \subseteq {}^\lambda 2$ for some $\lambda < \kappa$ because κ is regular, and since each sequence in T can be extended to some $s \in \partial T$, this contradicts the fact that T has height κ . If $g: \kappa \rightarrow \partial T$ is a bijection, the map

$$f: {}^\kappa \kappa \rightarrow {}^\kappa 2, \quad x \mapsto g(x(0)) \wedge g(x(1)) \wedge \dots \wedge g(x(\alpha)) \wedge \dots$$

²⁰To see that if κ is singular and not strong limit then $2^{<\kappa} > \kappa$, let $\text{cof}(\kappa) \leq \lambda < \kappa$ be such that $2^\lambda \geq \kappa$. Then $\kappa < \kappa^{\text{cof}(\kappa)} \leq (2^\lambda)^{\text{cof}(\kappa)} = 2^\lambda \leq 2^{<\kappa}$.

is a homeomorphism between $({}^\kappa\kappa, \tau_b)$ and $({}^\kappa 2, \tau_b)$, as required.

(b) If κ is not strong limit, then there is $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. Arguing as in part (a), we have that there is a homeomorphism $f: ({}^\kappa\kappa, \tau_b) \rightarrow ({}^\kappa 2, \tau_b)$. Conversely, assume that κ is strong limit, so that $2^{<\kappa} = \kappa$. If $\langle U_\alpha \mid \alpha < \lambda \rangle$ is a τ_b -open partition of ${}^\kappa 2$, then $\lambda \leq \kappa$: in fact, the map assigning to each $\alpha < \lambda$ some $s_\alpha \in {}^{<\kappa}2$ such that $N_{s_\alpha} \subseteq U_\alpha$ is injective and witnesses $\lambda \leq 2^{<\kappa} = \kappa$. Since $\{\widehat{N}_s \mid s \in \text{cof}(\kappa)^\kappa\}$ is a τ_b -open partition of ${}^\kappa\kappa$ of size $> \kappa$, this shows that there is no continuous surjection $f: ({}^\kappa 2, \tau_b) \rightarrow ({}^\kappa\kappa, \tau_b)$. \square

Proposition 6.6 shows that under AC the spaces $({}^\kappa 2, \tau_b)$ and $({}^\kappa\kappa, \tau_b)$ are in most cases homeomorphic, including e.g. when $\kappa = \omega_1$. The situation is quite different in models of determinacy. Assuming AD, the cardinal ω_1 is measurable, so it has the tree property. Moreover, $\omega_1 \not\rightarrow \omega_2$ by the PSP, and therefore $({}^{\omega_1}2, \tau_b)$ and $({}^{\omega_1}\omega_1, \tau_b)$ are not homeomorphic by Lemma 6.4(b). This argument can be generalized to larger *regular* cardinals: in fact, Steel and Woodin have shown that $\text{AD} + \text{V} = \text{L}(\mathbb{R})$ implies that every uncountable regular $\kappa < \Theta$ is measurable, and hence it has the tree property, and $\lambda^+ \not\rightarrow \lambda^2$ for all $\lambda < \Theta$ — see [Ste10, SWar]. Woodin (unpublished) has weakened the hypothesis $\text{AD} + \text{V} = \text{L}(\mathbb{R})$ to $\text{AD}^+ + \text{V} = \text{L}(\mathcal{P}(\mathbb{R}))$ — see Definition 9.24 for AD^+ . Therefore by the argument above we have that:

PROPOSITION 6.7. *Assume $\text{AD}^+ + \text{V} = \text{L}(\mathcal{P}(\mathbb{R}))$. Then for every regular $\kappa < \Theta$, the spaces $({}^\kappa 2, \tau_b)$ and $({}^\kappa\kappa, \tau_b)$ are not homeomorphic.*

Therefore AC is a necessary assumption for Proposition 6.6. In fact, Propositions 6.6 and 6.7 show that it is independent of $\text{ZF} + \text{DC}$ whether $({}^{\omega_1}2, \tau_b)$ is homeomorphic to $({}^{\omega_1}\omega_1, \tau_b)$.

6.2. Baire category.

DEFINITION 6.8. Let μ be an infinite cardinal. A subset A of a topological space (X, τ) is said to be **μ -meager** if it is the union of μ -many nowhere dense sets, where a set is nowhere dense if it is disjoint from some open dense subset of X . A set $A \subseteq X$ is said to be **μ -comeager** if its complement is μ -meager. If $U \subseteq X$ is a nonempty open set, then $A \subseteq X$ is **μ -meager in U** (respectively, **μ -comeager in U**) if $A \cap U$ is μ -meager (respectively, μ -comeager) in the (topological) space U endowed with the relative topology induced by τ .

Every subset of a μ -meager set is μ -meager as well. If $\mu \leq \mu'$ are infinite cardinal and $U \subseteq V$ are open sets of (X, τ) , then every set $A \subseteq X$ which is μ -(co)meager in V is also μ' -(co)meager in U .

DEFINITION 6.9. Let μ be an infinite cardinal. A topological space X is said to be **μ -Baire** if the intersection of μ -many open dense subsets of X is dense in X .

As for the classical notion of a Baire space (which corresponds to the case $\mu = \omega$), it is easy to check that the space X is μ -Baire if and only if every μ -comeager subset of X is dense, if and only if every nonempty open subset of X is not μ -meager. Moreover, if $\mu \leq \mu'$ then every μ' -Baire space is automatically μ -Baire, and every open subspace of a μ -Baire space is μ -Baire as well.

THEOREM 6.10. *Let κ, λ be cardinals such that $\omega < \lambda < \min(\text{cof}(\kappa)^+, \kappa)$.*

- (a) *The space $({}^\kappa\kappa, \tau_p)$ is ω -Baire.*
- (b) *Assume AC. Then the space $({}^\kappa\kappa, \tau_\lambda)$ is $\text{cof}(\lambda)$ -Baire.*
- (c) *Assume AC. Then the space $({}^\kappa\kappa, \tau_b)$ is $\text{cof}(\kappa)$ -Baire.*

All of (a)–(c) holds true when ${}^\kappa\kappa$ is replaced by ${}^\kappa 2$.

Theorem 6.10 can be restated (and easily proved) in the language of forcing. If \mathbf{P} is a forcing notion (i.e. a quasi-order), then $\text{FA}_\mu(\mathbf{P})$, the **μ -forcing axiom for \mathbf{P}** , says that for any sequence

$\langle D_\alpha \mid \alpha < \mu \rangle$ of dense subsets of \mathbf{P} there is a filter G intersecting all D_α 's. Recalling that the set of partial functions is a forcing notion with reverse inclusion (see Section 2.1.2), the preceding result asserts that certain forcing axioms hold:

- (a*) $\text{FA}_\omega(\mathbf{Fn}(\kappa, \kappa; \omega))$ and $\text{FA}_\omega(\mathbf{Fn}(\kappa, 2; \omega))$.
- (b*) Assume AC. Then $\text{FA}_{\text{cof}(\lambda)}(\mathbf{Fn}(\kappa, \kappa; \lambda))$ and $\text{FA}_{\text{cof}(\lambda)}(\mathbf{Fn}(\kappa, 2; \lambda))$.
- (c*) Assume AC. Then $\text{FA}_{\text{cof}(\kappa)}(\mathbf{Fn}(\kappa, \kappa; b))$ and $\text{FA}_{\text{cof}(\kappa)}(\mathbf{Fn}(\kappa, 2; b))$.

Theorem 6.10 cannot be extended to arbitrary closed subspaces of ${}^\kappa\kappa$, as shown by the next example.

EXAMPLE 6.11. Given a cardinal κ of uncountable cofinality, let $C \subseteq {}^\kappa\kappa$ be the collection of those $x \in {}^\kappa 2$ such that $|\{\alpha < \kappa \mid x(\alpha) = 0\}| < \omega$. Then C is τ_{ω_1} -closed, and hence also τ_b -closed and τ_λ -closed for every $\omega_1 \leq \lambda \leq \text{cof}(\kappa)$. We claim that C is not ω -Baire with respect to any of the topologies τ_p , τ_λ , or τ_b . For each $n \in \omega$, let $U_n := \{x \in {}^\kappa\kappa \mid |\{\alpha < \kappa \mid x(\alpha) = 0\}| \geq n\}$. Each U_n is open and dense with respect to any of the above topologies, and $U_n \cap C \neq \emptyset$. However $\bigcap_{n \in \omega} (U_n \cap C) = (\bigcap_{n \in \omega} U_n) \cap C = \emptyset$, and hence C is not ω -Baire.

For the results of this paper, we need to show that also $\text{Sym}(\kappa)$, which is a Π_2^0 -subset of ${}^\kappa\kappa$ with respect to any of our topologies, is μ -Baire for appropriate cardinals μ . However, since Example 6.11 shows that there may be even simple closed sets that are not ω -Baire, the μ -Baireness of $\text{Sym}(\kappa)$ needs to be proved independently of Theorem 6.10. To state the next result in the forcing jargon we need the following definition: for κ, λ infinite cardinals, let

$$\begin{aligned} \mathbf{Inj}(\kappa, \kappa; \lambda) &:= \{s \in \mathbf{Fn}(\kappa, \kappa; \lambda) \mid s \text{ is injective}\}, \\ \mathbf{Inj}(\kappa, \kappa; b) &:= \{s \in \mathbf{Fn}(\kappa, \kappa; b) \mid s \text{ is injective}\} = {}^{<\kappa}(\kappa). \end{aligned}$$

THEOREM 6.12. Let κ, λ be cardinals such that $\omega < \lambda < \min(\text{cof}(\kappa)^+, \kappa)$.

- (a) The space $(\text{Sym}(\kappa), \tau_p)$ is ω -Baire, i.e. $\text{FA}_\omega(\mathbf{Inj}(\kappa, \kappa; \omega))$.
- (b) Assume AC. Then the space $(\text{Sym}(\kappa), \tau_\lambda)$ is $\text{cof}(\lambda)$ -Baire, i.e. $\text{FA}_{\text{cof}(\lambda)}(\mathbf{Inj}(\kappa, \kappa; \lambda))$.
- (c) Assume AC. Then the space $(\text{Sym}(\kappa), \tau_b)$ is $\text{cof}(\kappa)$ -Baire, i.e. $\text{FA}_{\text{cof}(\kappa)}(\mathbf{Inj}(\kappa, \kappa; b))$.

At first glance, the forcing axioms in the second part of (a)–(c) may seem weaker than their counterparts for μ -Baireness of $\text{Sym}(\kappa)$ because in all the three cases an arbitrary generic intersecting a given family of μ -many dense subsets of the corresponding forcing poset may fail to be surjective. However, the arguments presented in the proof below show that the two formulations of each point are indeed equivalent: the existence of an arbitrary generic granted by the forcing axiom implies the existence of a surjective one, thus yielding the corresponding μ -Baireness property for $\text{Sym}(\kappa)$.

PROOF OF THEOREM 6.12. Let us first consider (b). The argument is essentially the same as the one used in the classical Baire category theorem (see e.g. [Kec95, Theorem 8.4]) — we present it here for the reader's convenience. Let $U \subseteq \text{Sym}(\kappa)$ be open and nonempty, and let $\langle U_\alpha \mid \alpha < \text{cof}(\lambda) \rangle$ be a sequence of open dense subsets of $\text{Sym}(\kappa)$. For every $\alpha < \text{cof}(\lambda)$, we recursively define $s_\alpha \in \mathbf{Inj}(\kappa, \kappa; \lambda)$ so that $\widehat{N}_{s_\alpha} \cap \text{Sym}(\kappa) \subseteq U \cap \bigcap_{\beta < \alpha} U_\beta$, and $s_\beta \subseteq s_\alpha$ for $\beta \leq \alpha < \text{cof}(\lambda)$. Let $s_0 \in \mathbf{Inj}(\kappa, \kappa; \lambda)$ be such that $\widehat{N}_{s_0} \cap \text{Sym}(\kappa) \subseteq U$. Let now $\alpha := \gamma + 1$: since U_α is open and dense in $\text{Sym}(\kappa)$ and $s_\gamma \in \mathbf{Inj}(\kappa, \kappa; \lambda)$, $U_\alpha \cap \widehat{N}_{s_\gamma}$ is nonempty and open in $\text{Sym}(\kappa)$. Pick $s' \in \mathbf{Inj}(\kappa, \kappa; \lambda)$ so that $\widehat{N}_{s'} \cap \text{Sym}(\kappa) \subseteq U_\alpha \cap \widehat{N}_{s_\gamma}$, and notice that s' is necessarily compatible with s_γ . Then $s_\alpha := s_\gamma \cup s'$ has the required properties. Finally, let $\alpha < \text{cof}(\lambda)$ be limit. Since all the s_β 's for $\beta < \alpha$ are compatible and belong to $\mathbf{Inj}(\kappa, \kappa; \lambda)$, $s_\alpha := \bigcup_{\beta < \alpha} s_\beta \in \mathbf{Inj}(\kappa, \kappa; \lambda)$ has the required properties. Let now $t := \bigcup_{\alpha < \text{cof}(\lambda)} s_\alpha$, so that t is an injective (partial) function from κ into itself. Since $|\text{dom}(t)| = |\text{ran}(t)| \leq \lambda < \kappa$, we can

pick a bijection t' between $\kappa \setminus \text{dom}(t)$ and $\kappa \setminus \text{ran}(t)$: then $x := t \cup t' \in \text{Sym}(\kappa)$, and x witnesses $U \cap \bigcup_{\alpha < \text{cof}(\lambda)} U_\alpha \neq \emptyset$.

(a) can be proved in a similar way, using the fact that $\kappa > \omega$ — just notice that when dealing with the product topology we do not need the Axiom of Choice to recursively pick the sequences s_α because $|\kappa|^{<\omega} = \kappa$ implies that $\widehat{\mathcal{B}}_p(\kappa)$ is well-orderable.

Finally, to prove (c) we use a sort of back-and-forth argument. Let $U \subseteq \text{Sym}(\kappa)$ be open and nonempty, and let $\langle U_\alpha \mid \alpha < \text{cof}(\kappa) \rangle$ be a sequence of open dense subsets of $\text{Sym}(\kappa)$. Fix an increasing sequence $\langle \lambda_\alpha \mid \alpha < \text{cof}(\kappa) \rangle$ of ordinals cofinal in κ . Let $s_0 \in {}^{<\kappa}\kappa$ be an injective sequence such that $\widehat{N}_{s_0} \cap \text{Sym}(\kappa) \subseteq U$, and for α a limit ordinal let $s_\alpha := \bigcup_{\beta < \alpha} s_\beta$, which is injective whenever all the s_β 's are injective. Let now γ be either 0 or a limit ordinal $< \text{cof}(\kappa)$. For $n \in \omega$ and $i \in \{1, 2\}$, we recursively define $s_{\gamma+2n+i}$ as follows. If $i = 1$ we let $s_{\gamma+2n+1} \in {}^{<\kappa}\kappa$ be any injective sequence extending $s_{\gamma+2n}$ such that $\widehat{N}_{s_{\gamma+2n+1}} \subseteq U_{\gamma+n}$ (which exists because the U_α 's are open and dense in $\text{Sym}(\kappa)$). If instead $i = 2$, let $\eta := \text{lh } s_{\gamma+2n+1}$, and let $\langle j_l \mid l < \delta \rangle$ be a strictly increasing enumeration of $\lambda_{\gamma+n} \setminus \text{ran}(s_{\gamma+2n+1})$ (for a suitable $\delta \leq \lambda_{\gamma+n}$). Let $s_{\gamma+2n+2}$ be the extension of $s_{\gamma+2n+1}$ of length $\eta + \delta$ obtained by setting $s_{\gamma+2n+2}(\eta + l) := j_l$ for every $l < \delta$, so that $s_{\gamma+2n+2}$ is still injective and $\lambda_{\gamma+2n} \subseteq \text{ran}(s_{\gamma+2n+2})$. Then it is easy to check that $x := \bigcup_{\alpha < \text{cof}(\kappa)} s_\alpha \in \text{Sym}(\kappa)$ and $x \in U \cap \bigcup_{\alpha < \text{cof}(\lambda)} U_\alpha$. \square

The following is a variant of the classical Baire property, corresponding to the case $\mu = \omega$.

DEFINITION 6.13. Let μ be an infinite cardinal and X be a topological space. We say that $A \subseteq X$ has the μ -**Baire property** if there is an open set $U \subseteq X$ such that the symmetric difference $A \triangle U$ is μ -meager.

As shown in [HS01], arguing as in the classical case one easily gets the following results (see also e.g. [Kec95, Proposition 8.22] and [Kec95, Proposition 8.26]).

PROPOSITION 6.14. *Let μ be an infinite cardinal and X be a topological space. The class of all subsets of X having the μ -Baire property is a $\mu + 1$ -algebra on X , and in fact it is the $\mu + 1$ -algebra on X generated by all open sets and all μ -meager sets. In particular, all sets in $\mathcal{B}_{\mu+1}(X)$ have the μ -Baire property.*

PROPOSITION 6.15. *Let μ be an infinite cardinal and let X be a topological space. If $A \subseteq X$ has the μ -Baire property and is not μ -meager, then there is a nonempty open set $U \subseteq X$ such that A is μ -comeager in U .*

7. Standard Borel κ -spaces, κ -analytic quasi-orders, and spaces of codes

7.1. κ -analytic sets. Recall that a subset A of a Polish space X is called analytic if it is a continuous image of a closed subset of the Baire space ${}^\omega\omega$. Here are some reformulations of this notion, where p denotes the projection on the first coordinate, as defined in (2.8):

- $A \subseteq X$ is analytic if and only if it is either empty or a continuous image of the whole Baire space ${}^\omega\omega$;
- $A \subseteq X$ is analytic if and only if it is a continuous image of a Borel subset of ${}^\omega\omega$;
- $A \subseteq X$ is analytic if and only if $A = p F$ for some closed $F \subseteq X \times {}^\omega\omega$;
- $A \subseteq X$ is analytic if and only if $A = p B$ for some Borel $B \subseteq X \times {}^\omega\omega$.

It is not hard to see that the class of analytic sets contains all Borel sets and is closed under countable unions, countable intersections, and images and preimages under Borel functions. In Section 9 the collection of analytic sets will be identified with $\mathcal{S}(\omega)$, the class of ω -Souslin sets.

We now generalize the notion of analytic set to the uncountable context. To simplify the presentation, in the subsequent definition and results we endow the generalized Baire space

${}^\kappa\kappa$ with the bounded topology, so that all the related topological notions (such as continuous functions, $\kappa + 1$ -Borel sets, and so on) tacitly refer to τ_b . Of course analogous notions can be obtained by replacing, *mutatis mutandis*, the topology τ_b with any of the topologies introduced in Definition 6.1. However, since we have no use for these variants in the rest of the paper, for the sake of simplicity we leave to the reader the burden of checking which of the properties stated below transfer to these topologies.

DEFINITION 7.1. A set $A \subseteq {}^\kappa\kappa$ is called **κ -analytic** if it is a continuous image of a closed subset of ${}^\kappa\kappa$.

Unlike in the classical case $\kappa = \omega$, when $\kappa > \omega$ it is no more true in general that nonempty κ -analytic sets are continuous images of the whole ${}^\kappa\kappa$ — see [LS15]. However, all the other equivalent reformulations mentioned above remain true also in our new context. The key result to prove this is the following proposition.

PROPOSITION 7.2. *Every effective $\kappa + 1$ -Borel subset of ${}^\kappa\kappa$ is κ -analytic.*

PROOF. We modify the proof of [MR13, Lemma 3.9 and Proposition 3.10], where the desired result is proved under²¹ AC and the extra cardinal assumption $\kappa^{<\kappa} = \kappa$. In particular, in that proof it is argued in ZF that given a family $\{C_\alpha \mid \alpha < \kappa\}$ of closed subsets of ${}^\kappa\kappa$ and a corresponding family of continuous maps $f_\alpha: C_\alpha \rightarrow {}^\kappa\kappa$ with range A_α , one can canonically construct two closed sets $C_\cap, C_\cup \subseteq {}^\kappa\kappa$ and continuous maps $f_\cap: C_\cap \rightarrow {}^\kappa\kappa$ and $f_\cup: C_\cup \rightarrow {}^\kappa\kappa$ such that f_\cap surjects onto $\bigcap_{\alpha < \kappa} A_\alpha$ and f_\cup surjects onto $\bigcup_{\alpha < \kappa} A_\alpha$. Moreover, the identity function witnesses that every closed subset of ${}^\kappa\kappa$ is κ -analytic. We now show (in ZF, and without assuming $\kappa^{<\kappa} = \kappa$) that also all open sets are κ -analytic. Let $U \subseteq {}^\kappa\kappa$ be open, and for every $\alpha < \kappa$ set $S_\alpha := \{s \in {}^\alpha\kappa \mid \widehat{N}_s^\kappa \subseteq U\}$, so that $U = \bigcup_{\alpha < \kappa} U_\alpha$ with $U_\alpha := \bigcup_{s \in S_\alpha} \widehat{N}_s^\kappa$. Notice that each U_α is clopen, and therefore κ -analytic. Setting $C_\alpha := U_\alpha$ and $f_\alpha := \text{id} \upharpoonright U_\alpha$, we get that the map $f_\cup: C_\cup \rightarrow \bigcup_{\alpha < \kappa} U_\alpha \subseteq {}^\kappa\kappa$ as above witnesses that U is κ -analytic.

Let now $B \subseteq {}^\kappa\kappa$ be effective $\kappa + 1$ -Borel and let (T, ϕ) be a $\kappa + 1$ -Borel code for it. Using the facts mentioned in the previous paragraph, one can easily build by recursion on the rank of the nodes of the well-founded tree T a map g on T assigning to each $t \in T$ two closed sets $C_t, C'_t \subseteq {}^\kappa\kappa$ and two continuous functions $f_t: C_t \rightarrow {}^\kappa\kappa$ and $f'_t: C'_t \rightarrow {}^\kappa\kappa$ such that f_t surjects onto $\phi(t)$ and f'_t surjects onto ${}^\kappa\kappa \setminus \phi(t)$. In particular, C_\emptyset and f_\emptyset witness that B is κ -analytic. \square

The proof of Proposition 7.2 also implies that the class of κ -analytic subsets of ${}^\kappa\kappa$ is closed under finite intersections and finite unions. It is also closed under intersections and unions of length $\alpha \leq \kappa$ as long as given a family of κ -analytic sets $\{A_\beta \mid \beta < \alpha\}$ one can choose witnesses $f_\beta: C_\beta \rightarrow A_\beta$ of this. Therefore if we assume AC_κ the class of κ -analytic subsets of ${}^\kappa\kappa$ is closed under intersections and unions of length $\leq \kappa$.

COROLLARY 7.3. *The following are equivalent for a set $A \subseteq {}^\kappa\kappa$:*

- (a) A is κ -analytic;
- (b) A is a continuous image of an effective $\kappa + 1$ -Borel subset of ${}^\kappa\kappa$;
- (c) $A = pF$ for some closed $F \subseteq {}^\kappa\kappa \times {}^\kappa\kappa$;
- (d) $A = pB$ for some effective $\kappa + 1$ -Borel $B \subseteq {}^\kappa\kappa \times {}^\kappa\kappa$.

PROOF. (a) implies (b) and (c) implies (d) because every closed set is effective $\kappa + 1$ -Borel. Moreover, (b) implies (a) by Proposition 7.2, and (d) implies (b) because ${}^\kappa\kappa \times {}^\kappa\kappa$ is homeomorphic to ${}^\kappa\kappa$ and the projection map is continuous. So it is enough to show that (a) implies (c). Let $f: C \rightarrow {}^\kappa\kappa$ be a continuous surjection onto A with $C \subseteq {}^\kappa\kappa$ closed. Since ${}^\kappa\kappa$ is an Hausdorff space, this implies that the graph F of f is closed in $C \times {}^\kappa\kappa$, and hence also closed in ${}^\kappa\kappa \times {}^\kappa\kappa$. As $A = p(F^{-1})$ the result is proved. \square

²¹Under choice all $\kappa + 1$ -Borel sets are effective.

One of the main use of Corollary 7.3 is that it allows us to use (a generalization of) the Tarski-Kuratowski algorithm (see e.g. [Kec95, Appendix C]) to establish that a given set $A \subseteq {}^\kappa\kappa$ is κ -analytic by inspecting the “logical form” of its definition.

The definition of κ -analyticity can be extended to subsets of an arbitrary subspace S of ${}^\kappa\kappa$: $A \subseteq S$ is κ -analytic (in S) if and only if there is a κ -analytic subset A' of ${}^\kappa\kappa$ such that $A = A' \cap S$. Notice that if S is an effective $\kappa + 1$ -Borel subset of ${}^\kappa\kappa$, then by Proposition 7.2 and the observation following it we get that $A \subseteq S$ is κ -analytic in S if and only if it is κ -analytic in the whole ${}^\kappa\kappa$. This allows us to have a natural definition of κ -analytic subset for any space which is effectively $\kappa + 1$ -Borel isomorphic to some $B \in \mathbf{B}_{\kappa+1}^e({}^\kappa\kappa, \tau_b)$, leading to the following definitions which generalize the case $\kappa = \omega$.

DEFINITION 7.4. Let $\mathcal{B} \subseteq \mathcal{P}(X)$ be an algebra on a nonempty set X . We say that (X, \mathcal{B}) is a **standard Borel κ -space**²² if there is a topology τ on X such that $\mathcal{B} = \mathbf{B}_{\kappa+1}^e(X, \tau)$ and (X, τ) is homeomorphic to an effective $\kappa + 1$ -Borel subset of ${}^\kappa\kappa$. If the algebra \mathcal{B} is clear from the context, we say that X is a standard Borel κ -space.

The collection of standard Borel κ -spaces is closed under effective $\kappa + 1$ -Borel subsets, that is: if (X, \mathcal{B}) is a standard Borel κ -space then for every $B \in \mathcal{B}$ the space $(B, \mathcal{B} \upharpoonright B)$ is a standard Borel κ -space as well, where $\mathcal{B} \upharpoonright B := \{B' \cap B \mid B' \in \mathcal{B}\}$. In particular, for every $B \in \mathbf{B}_{\kappa+1}^e({}^\kappa\kappa, \tau_b)$ the space

$$(B, \mathbf{B}_{\kappa+1}^e({}^\kappa\kappa, \tau_b) \upharpoonright B) = (B, \mathbf{B}_{\kappa+1}^e(B, \tau_b))$$

is a standard Borel κ -space. Moreover, the product and the disjoint union of finitely many standard Borel κ -spaces are again standard Borel κ -spaces (where a product $X \times X'$ is equipped with the product $\mathcal{B} \otimes \mathcal{B}'$ of the algebras \mathcal{B} and \mathcal{B}' on X and X' , and a disjoint union $X \uplus X'$ is equipped with the corresponding union algebra $\mathcal{B} \oplus \mathcal{B}'$). Assuming enough choice (AC_κ suffices), the same is true for products and unions of length $\leq \kappa$.

Notice that Definitions 7.5 and 7.7 are independent from the choice of the witness that X is a standard Borel κ -space.

DEFINITION 7.5. Let X be a standard Borel κ -space. A set $A \subseteq X$ is **κ -analytic** if for any topology τ , any set $B \in \mathbf{B}_{\kappa+1}^e({}^\kappa\kappa, \tau_b)$, and any homeomorphism $f: (X, \tau) \rightarrow (B, \tau_b \upharpoonright B)$ witnessing that X is standard Borel, the set $f(A)$ is a κ -analytic subset of ${}^\kappa\kappa$ (equivalently, of B).

The above definition directly implies that analogues of Proposition 7.2 and Corollary 7.3 hold in the broader context of standard Borel κ -spaces.

PROPOSITION 7.6. Let (X, \mathcal{B}) be a standard Borel κ -space.

- (a) Every $B \in \mathcal{B}$ is κ -analytic.
- (b) The following are equivalent for $A \subseteq X$:
 - A is κ -analytic;
 - A is a continuous image of an effective $\kappa + 1$ -Borel subset of ${}^\kappa\kappa$ (where continuity refers to any topology τ on X witnessing that it is a standard Borel κ -space);
 - $A = pF$ for some closed $F \subseteq X \times {}^\kappa\kappa$ (where X is endowed with any τ as above);
 - $A = pB$ for some $B \in \mathcal{B} \otimes \mathbf{B}_{\kappa+1}^e({}^\kappa\kappa, \tau_b)$.

DEFINITION 7.7. A **κ -analytic quasi-order** S on a standard Borel κ -space X is a quasi-order on X which is κ -analytic as a subset of $X \times X$. If moreover S is symmetric, then it is called a **κ -analytic equivalence relation** (on X).

Examples of natural κ -analytic quasi-orders and equivalence relations are given in Section 7.2.

²²Our definition of a standard Borel κ -space slightly differs from the one introduced in [MR13, Definition 3.6]; however, the two definitions essentially coincide in the setup of [MR13], i.e. when assuming AC and $\kappa^{<\kappa} = \kappa$.

7.2. Spaces of type κ and spaces of codes. In this section the notion of a space of type κ is defined — these are spaces which are homeomorphic to ${}^\kappa 2$ in a canonical way. This notion is introduced to ease the study of spaces of codes for \mathcal{L} -structures of size κ for some finite relational language \mathcal{L} , for complete metric spaces of density character κ , and for Banach spaces of density κ . All spaces of type κ and their effective $\kappa + 1$ -Borel subsets are also standard Borel κ -spaces when equipped with the algebra of their effective $\kappa + 1$ -Borel subsets.

7.2.1. *Spaces of type κ .* Let A be a set of size κ . Then any bijection $f: \kappa \rightarrow A$ induces a bijection between ${}^\kappa 2$ and ${}^A 2$, so the product topology, the λ -topology (for $\omega \leq \lambda < \max(\text{cof}(\kappa)^+, \kappa)$), and the bounded topology can be copied on ${}^A 2$, and are denoted with $\tau_p({}^A 2)$, $\tau_\lambda({}^A 2)$, and $\tau_b({}^A 2)$, respectively. The bases for these topologies are given by

$$\begin{aligned}\mathcal{B}_p({}^A 2) &:= \{N_s^A \mid |s| < \omega\} \\ \mathcal{B}_\lambda({}^A 2) &:= \{N_s^A \mid |s| < \lambda\} \\ \mathcal{B}_b({}^A 2) &:= \{N_s^A \mid \exists \alpha < \kappa (f \restriction \alpha = \text{dom } s)\}\end{aligned}$$

where

$$N_s^A := \{x \in {}^A 2 \mid s \subseteq x\}$$

for s a partial function from A to 2 . Note that $\mathcal{B}_p({}^A 2) = \mathcal{B}_\omega({}^A 2)$, since $\tau_p({}^A 2) = \tau_\omega({}^A 2)$. As in Remark 3.2(ii), the collection of all

$$(7.1) \quad \tilde{N}_{a,i}^A := \{x \in {}^A 2 \mid x(a) = i\} \quad (a \in A, i \in \{0, 1\})$$

is a subbasis (generating $\mathcal{B}_p({}^A 2)$) for $\tau_p({}^A 2)$.

The definitions of $\tau_\lambda({}^A 2)$ and $\mathcal{B}_\lambda({}^A 2)$ (which includes the case of the product topology) are independent of the chosen f . The situation for $\tau_b({}^A 2)$ is rather different — its definition is again independent of f when κ is regular, but this is no more true when κ is singular. Moreover, the canonical basis $\mathcal{B}_b({}^A 2)$ always depend on f , even when κ is regular. This is an unpleasant situation; however, in our applications this will not be an issue, as there will always be a *canonical* bijection between A and κ . To illustrate this, let us consider two representative examples.

EXAMPLES 7.8. (A) Consider the set $A := {}^{<\omega} 2 \times \kappa$ (the topological space $({}^A 2, \tau_p)$ plays an important role in Section 12). Let $\theta: {}^{<\omega} 2 \rightarrow \omega$ be the unique isomorphism between $({}^{<\omega} 2, \preceq)$ and (ω, \leq) , where \preceq is defined by

$$u \preceq v \Leftrightarrow \text{lh } u < \text{lh } v \vee (\text{lh } u = \text{lh } v \wedge u \leq_{\text{lex}} v)$$

with \leq_{lex} the usual lexicographical order. Then

$$f: {}^{<\omega} 2 \times \kappa \rightarrow \kappa, \quad (u, \alpha) \mapsto \omega \cdot \alpha + \theta(u)$$

can be taken to be a standard bijection between A and κ , as it orders A antilexicographically. With this bijection the notions of “boundness” and “initial segment” on A become natural and unambiguous.

(B) Recall from (2.1) the standard pairing function for ordinals $\langle \cdot, \cdot \rangle: \text{Ord} \times \text{Ord} \rightarrow \text{Ord}$. For $n \geq 2$ define the bijections $f_n: {}^n \kappa \rightarrow \kappa$ by setting

$$\begin{aligned}f_2(\alpha_0, \alpha_1) &:= \langle \alpha_0, \alpha_1 \rangle, \\ f_{n+1}(\alpha_0, \dots, \alpha_n) &:= \langle f_n(\alpha_0, \dots, \alpha_{n-1}), \alpha_n \rangle.\end{aligned}$$

Each f_n can be considered as the standard bijection between $A := {}^n \kappa$ and κ .

The next definition tries to capture the content of Examples 7.8.

DEFINITION 7.9. Let κ be an infinite cardinal. A **space of type κ** is a set of the form $\mathcal{X} = \prod_{i \in I} {}^{A_i}2$, where I is a finite set and each A_i is a set of cardinality κ .

The bounded topology $\tau_b(\mathcal{X})$ on \mathcal{X} is the product of the bounded topologies $\tau_b({}^{A_i}2)$ on ${}^{A_i}2$, and is generated by the canonical basis

$$\mathcal{B}_b^{\mathcal{X}} := \{\prod_{i \in I} N_{s_i}^{A_i} \mid N_{s_i}^{A_i} \in \mathcal{B}_b({}^{A_i}2) \text{ for all } i \in I\}.$$

Similarly, $\tau_p(\mathcal{X})$ and $\tau_\lambda(\mathcal{X})$ are defined as the product of the corresponding topologies on each factor, and their bases $\mathcal{B}_p(\mathcal{X})$ and $\mathcal{B}_\lambda(\mathcal{X})$ are defined as the products of the corresponding bases.

All definitions, observations, and results concerning ${}^\kappa 2$ considered in Sections 3–5 can be applied to an arbitrary space \mathcal{X} of type κ as well, including the following:

- The collection $\mathbf{B}_\alpha(\mathcal{X}, \tau_*) = \mathbf{B}_\alpha(\tau_*)$ of α -Borel subsets of \mathcal{X} , where τ_* is one of the topologies $\tau_p(\mathcal{X})$, $\tau_\lambda(\mathcal{X})$, or $\tau_b(\mathcal{X})$ (Definition 4.1), together with its effective counterpart $\mathbf{B}_\alpha^e(\mathcal{X}, \tau_*)$ (Section 4.1.2).
- The (weakly) α -Borel functions $f: \mathcal{Y} \rightarrow \mathcal{Z}$, where \mathcal{Y} and \mathcal{Z} are arbitrary spaces of type λ and μ , respectively (Definition 5.1).
- The Γ -in-the-codes function $f: {}^\omega 2 \rightarrow \mathcal{X}$ for Γ a suitable boldface pointclass (Definition 5.3).
- The fact that the collection of all $\tau_b(\mathcal{X})$ -clopen subsets is a $\text{cof}(\kappa)$ -algebra on \mathcal{X} (Proposition 3.12(j)).
- The fact that assuming AC, if $2^{<\kappa} = \kappa$ then $\mathbf{B}_{\kappa+1}(\tau_p(\mathcal{X})) = \mathbf{B}_{\kappa+1}(\tau_b(\mathcal{X}))$ (Corollary 4.10), and therefore $\mathbf{B}_{\kappa+1}(\tau_\lambda(\mathcal{X})) = \mathbf{B}_{\kappa+1}(\tau_b(\mathcal{X}))$ for every $\omega \leq \lambda < \max(\text{cof}(\kappa)^+, \kappa)$.

REMARK 7.10. We often consider (weakly) α -Borel functions $f: A \rightarrow B$ (for suitable ordinals α) where A and B are arbitrary subspaces of two spaces \mathcal{Y} and \mathcal{Z} of type λ and μ , respectively, each endowed with τ_b . By this we mean that f is (weakly) \mathcal{S} -measurable when B is endowed with the relative topologies inherited from \mathcal{Z} and \mathcal{S} is the algebra of subsets of A consisting of the traces on A of the α -Borel subsets of \mathcal{Y} . Notice that:

- (i) When $A \in \mathbf{B}_\alpha(\mathcal{Y})$, the notion of a (weakly) α -Borel function $f: A \rightarrow \mathcal{Z}$ is unambiguous since a set $C \subseteq A$ is α -Borel in A if and only if it is α -Borel as a subset of the entire space \mathcal{Y} . In fact, a function $f: A \rightarrow \mathcal{Z}$ is (weakly) α -Borel if and only if $f = g \upharpoonright A$ for some (weakly) α -Borel function $g: \mathcal{Y} \rightarrow \mathcal{Z}$.
- (ii) A function $f: \mathcal{Y} \rightarrow B$ is (weakly) α -Borel if and only if it is (weakly) α -Borel as a function between \mathcal{Y} and \mathcal{Z} .

Notice also that any space \mathcal{X} of type κ is, by definition, homeomorphic to ${}^\kappa 2$, which is a closed (and hence effective $\kappa + 1$ -Borel) subset of ${}^\kappa \kappa$. Thus \mathcal{X} equipped with the algebra of its effective $\kappa + 1$ -Borel subsets is also a standard Borel κ -space, and therefore we can consider the notions of a κ -analytic subset of \mathcal{X} and of a κ -analytic quasi-order on \mathcal{X} (or on one of its effective $\kappa + 1$ -Borel subsets) as in Definitions 7.5 and 7.7.

7.2.2. *Space of codes for (\mathcal{L}) -structures of size κ .* For I a finite set, let

$$\mathcal{L} = \{R_i \mid i \in I\}$$

be a relational²³ signature. For the sake of definiteness assume that $I \in \omega$ and that n_i is the arity of R_i , for $i \in I$. A **structure** or **model** for \mathcal{L} , sometimes called simply \mathcal{L} -structure, is an object of the form $\mathcal{A} = \langle A; R_i^{\mathcal{A}} \rangle_{i \in I}$, where A is a nonempty set and $R_i^{\mathcal{A}}$ is the interpretation of the symbol R_i in \mathcal{A} . If $\emptyset \neq B \subseteq A$ then we denote by $\mathcal{A} \upharpoonright B$ the restriction of \mathcal{A} to its subdomain B , i.e. $\mathcal{A} \upharpoonright B = \langle B; R_i^{\mathcal{A}} \cap {}^{n_i} B \rangle_{i \in I}$. When there is no danger of confusion, with abuse of notation the structure \mathcal{A} is identified with its domain A . Since the nature of the elements of

²³To simplify the presentation we retreat to relational signatures. This is not restrictive since any n -ary function may be identified with its graph, which is a relation of arity $n + 1$.

A is irrelevant, a model of size κ is taken to have domain κ , so that each R_i^A can be identified with its characteristic function ${}^{n_i}\kappa \rightarrow 2 = \{0, 1\}$. Therefore any model of size κ can be identified, up to isomorphism, with a map

$$\bigcup_{i \in I} \{i\} \times {}^{n_i}\kappa \rightarrow 2,$$

and hence

$$(7.2) \quad \text{Mod}_{\mathcal{L}}^{\kappa} := \prod_{i \in I} ({}^{n_i}\kappa)2$$

can be regarded as the **space of** (codes for) **all \mathcal{L} -structures of size κ** (up to isomorphism). We also set

$$\text{Mod}_{\mathcal{L}}^{<\kappa} := \bigcup_{\lambda < \kappa} \text{Mod}_{\mathcal{L}}^{\lambda} \quad \text{and} \quad \text{Mod}_{\mathcal{L}}^{\infty} := \bigcup_{\kappa \in \text{Card}} \text{Mod}_{\mathcal{L}}^{\kappa}.$$

For example, if \mathcal{L} is a relational language consisting of just one relational symbol of arity n , then $\text{Mod}_{\mathcal{L}}^{\kappa} = ({}^n\kappa)2$, and hence by Example 7.8(B) it can be topologized by τ_p , τ_{λ} (for $\omega \leq \lambda < \max(\text{cof}(\kappa)^+, \kappa)$), or τ_b . For richer languages $\mathcal{L} = \{R_i \mid i \in I\}$, the space of models $\text{Mod}_{\mathcal{L}}^{\kappa}$ in (7.2) is just a finite product of spaces of the form $({}^n\kappa)2$: thus it is a space of type κ , and hence a standard Borel κ -space.

Strictly speaking an $X \in \text{Mod}_{\mathcal{L}}^{\kappa}$ is a *function*, but it is more convenient to think of it as an \mathcal{L} -structure

$$X = \langle \kappa; R_i^X \rangle_{i \in I}.$$

The **embeddability** relation \sqsubset on $\text{Mod}_{\mathcal{L}}^{\kappa}$ is given by

$$(7.3) \quad X \sqsubset Y \Leftrightarrow \exists g \in {}^{\kappa}(\kappa) \forall i \in I \forall \langle x_1, \dots, x_{n_i} \rangle \in {}^{n_i}\kappa \quad [\langle x_1, \dots, x_{n_i} \rangle \in R_i^X \Leftrightarrow \langle g(x_1), \dots, g(x_{n_i}) \rangle \in R_i^Y]$$

where ${}^{\kappa}(\kappa)$ is the set of injective functions from κ into κ (see (2.3)). The equivalence relation induced by the quasi-order \sqsubset is the **bi-embeddability** relation \approx . If ${}^{\kappa}(\kappa)$ in (7.3) is replaced by $\text{Sym}(\kappa)$, the group of all bijections from κ to κ , the **isomorphism** relation \cong is obtained. Thus \cong can be seen as induced by the continuous action

$$(7.4) \quad \text{Sym}(\kappa) \times \text{Mod}_{\mathcal{L}}^{\kappa} \rightarrow \text{Mod}_{\mathcal{L}}^{\kappa}, \quad (g, X) \mapsto g.X$$

where $g.X \in \text{Mod}_{\mathcal{L}}^{\kappa}$ is defined by

$$(7.5) \quad R_i^{g.X} := \{ \langle y_1, \dots, y_{n_i} \rangle \in {}^{n_i}\kappa \mid \langle g^{-1}(y_1), \dots, g^{-1}(y_{n_i}) \rangle \in R_i^X \} \quad (i \in I).$$

The embeddability relation \sqsubset on $\text{Mod}_{\mathcal{L}}^{\kappa}$ is an example of a κ -analytic quasi-order. To see this, observe that ${}^{\kappa}(\kappa)$ is a closed (and hence effective $\kappa + 1$ -Borel) subset of ${}^{\kappa}\kappa$, so that \sqsubset is the projection on $\text{Mod}_{\mathcal{L}}^{\kappa} \times \text{Mod}_{\mathcal{L}}^{\kappa}$ of a closed subset of $\text{Mod}_{\mathcal{L}}^{\kappa} \times \text{Mod}_{\mathcal{L}}^{\kappa} \times {}^{\kappa}\kappa$ by (7.3). It follows that \approx is a κ -analytic equivalence relation on the standard Borel κ -space $\text{Mod}_{\mathcal{L}}^{\kappa}$. Similarly, using the fact that $\text{Sym}(\kappa)$ is an effective $\kappa + 1$ -Borel subset of ${}^{\kappa}\kappa$ (in fact: an effective intersection of κ -many open sets) one sees that \cong is a κ -analytic equivalence relation on $\text{Mod}_{\mathcal{L}}^{\kappa}$.

7.2.3. Space of codes for complete metric spaces of density character κ . In the classical case $\kappa = \omega$, there are essentially two ways for coding separable complete metric spaces, usually called **Polish metric spaces**, as elements of a standard Borel space.

The first, and more common one [CGK01, GK03, LR05], uses the fact that the (separable) Urysohn space \mathbb{U} is universal for this class, that is to say: \mathbb{U} is itself a Polish metric space (so that all its closed subsets are Polish metric spaces as well), and every Polish metric space is isometric to some closed subset of \mathbb{U} . Thus the space $F(\mathbb{U})$ of all closed subsets of \mathbb{U} endowed with its Effros-Borel structure, which is standard Borel (see e.g. [Kec95, Section 12.C]), may be regarded as a space of codes for all separable complete metric spaces. If this approach is to be generalized to an uncountable κ , a space which is universal for complete metric spaces of density character κ must be constructed. By [Kat88] analogues of the Urysohn space for larger density characters may be obtained only assuming AC and for cardinals κ satisfying $\kappa^{<\kappa} = \kappa$. Thus this

technique for coding metric spaces cannot be used here, since we want to study also choice-less models (such as models of AD), and even in the AC context we are interested in cardinals smaller than the continuum. To the best of our knowledge, there are no other kinds of universal spaces for complete metric spaces of density character κ in the literature, so we are forced to drop this approach.

The second way to code Polish metric spaces is to identify each of them with any of its dense subspaces, so that the original space may be recovered, up to isometry, as the completion of such a subspace (see e.g. [Ver98, Cle12]). Fortunately, this approach does generalize in ZF to any infinite cardinal κ , naturally yielding to the set of codes \mathfrak{M}_κ described below. Notice that the space of codes \mathfrak{M}_κ , being an effective $\kappa + 1$ -Borel subset of a space of type κ , carries a natural topology τ_b , and its effective $\kappa + 1$ -Borel structure turns it into a standard Borel κ -space.

Let \mathbb{Q}^+ be the set of positive rational numbers, and let \mathcal{X} be the space of type κ defined by $\mathcal{X} := {}^{\kappa \times \kappa \times \mathbb{Q}^+} 2$. Given a complete metric space (M, d) of density character κ and a dense subset $D = \{m_\alpha \mid \alpha < \kappa\}$ of it, we can identify M with the unique element $x_M \in \mathcal{X}$ such that for all $\alpha, \beta < \kappa$ and $q \in \mathbb{Q}^+$

$$(7.6) \quad x_M(\alpha, \beta, q) = 1 \Leftrightarrow d_M(m_\alpha, m_\beta) < q.$$

In fact, M is isometric to the completion of the metric space (κ, d_{x_M}) where $d_{x_M}(\alpha, \beta) := \inf\{q \in \mathbb{Q}^+ \mid x_M(\alpha, \beta, q) = 1\}$ for $\alpha, \beta < \kappa$, so that $d_{x_M}(\alpha, \beta) = d_M(m_\alpha, m_\beta)$ for all $\alpha, \beta < \kappa$. Consider now the space $\mathfrak{M}_\kappa \subseteq \mathcal{X}$ consisting of those $x \in {}^{\kappa \times \kappa \times \mathbb{Q}^+} 2$ satisfying the following conditions:

$$\begin{aligned} & \forall \alpha, \beta < \kappa \forall q, q' \in \mathbb{Q}^+ [q \leq q' \Rightarrow x(\alpha, \beta, q) \leq x(\alpha, \beta, q')] \\ & \forall \alpha, \beta < \kappa \exists q \in \mathbb{Q}^+ [x(\alpha, \beta, q) = 1] \\ & \forall \alpha < \kappa \forall q \in \mathbb{Q}^+ [x(\alpha, \alpha, q) = 1] \\ & \forall \alpha < \beta < \kappa \exists q \in \mathbb{Q}^+ [x(\alpha, \beta, q) = 0] \\ & \forall \alpha, \beta < \kappa \forall q \in \mathbb{Q}^+ [x(\alpha, \beta, q) = 1 \Leftrightarrow x(\beta, \alpha, q) = 1] \\ & \forall \alpha, \beta, \gamma < \kappa \forall q, q' \in \mathbb{Q}^+ [x(\alpha, \beta, q) = 1 \wedge x(\beta, \gamma, q') = 1 \Rightarrow x(\alpha, \gamma, q + q') = 1] \\ & \forall \alpha < \kappa \exists \beta < \kappa \exists q \in \mathbb{Q}^+ \forall \gamma < \alpha [x(\gamma, \beta, q) = 0]. \end{aligned}$$

The first six conditions are designed so that given any $x \in \mathfrak{M}_\kappa$, the (well-defined) map $d_x : \kappa \times \kappa \rightarrow \mathbb{R}$ defined by setting

$$d_x(\alpha, \beta) := \inf\{q \in \mathbb{Q}^+ \mid x(\alpha, \beta, q) = 1\}$$

is a metric on κ ; denote by M_x the completion of (κ, d_x) , and notice that the last condition ensures that M_x has density character κ . It is straightforward to check that the code x_M from (7.6) of any complete metric space M of density character κ belongs to \mathfrak{M}_κ , and is such that M is isometric to M_{x_M} ; conversely, for each $x \in \mathfrak{M}_\kappa$ the space M_x is a complete metric space of density character κ . Moreover, the explicit definition given above shows that $\mathfrak{M}_\kappa \in \mathbf{B}_{\kappa+1}^e(\mathcal{X}, \tau_b)$, so that \mathfrak{M}_κ is a standard Borel κ -space. Thus we can regard $\mathfrak{M}_\kappa \subseteq {}^{\kappa \times \kappa \times \mathbb{Q}^+} 2$, endowed with the inherited topologies and the corresponding (effective) $\kappa + 1$ -Borel structure, as the **space of** (codes for) **all complete metric spaces of density character κ** (up to isometry).

The **isometric embeddability** relation \sqsubseteq^i on \mathfrak{M}_κ is given by

$$x \sqsubseteq^i y \Leftrightarrow \text{there is a metric-preserving map from } M_x \text{ into } M_y,$$

while the **isometry** relation \cong^i on \mathfrak{M}_κ is given by

$$x \cong^i y \Leftrightarrow \text{there is a metric-preserving bijection between } M_x \text{ and } M_y.$$

Notice that for every $x, y \in \mathfrak{M}_\kappa$ one has

$$x \sqsubseteq^i y \Leftrightarrow \text{there is a metric-preserving map } i: (\kappa, d_x) \rightarrow M_y.$$

This allows us to check, using the Tarski-Kuratowski algorithm and some standard computations, that the relation \sqsubseteq^i is a κ -analytic quasi-order on \mathfrak{M}_κ , and a similar observation shows that \cong^i is a κ -analytic equivalence relation on \mathfrak{M}_κ . (For a blueprint of such computations, see [Cle12, Lemma 4].)

We consider some natural subclasses of \mathfrak{M}_κ , such as

$$\mathfrak{D}_\kappa := \{x \in \mathfrak{M}_\kappa \mid M_x \text{ is discrete}\}$$

or

$$\mathfrak{U}_\kappa := \{x \in \mathfrak{M}_\kappa \mid M_x \text{ is ultrametric}\}.$$

Notice however that not all these subclasses are standard Borel: for example, \mathfrak{U}_κ is a standard Borel κ -space (since it is a closed subset of \mathfrak{M}_κ), while \mathfrak{D}_κ is not.

7.2.4. Space of codes for Banach spaces of density κ . In analogy with what was done in Section 7.2.3 for complete metric spaces of density character κ , we code Banach spaces of density κ by identifying each of them with any of its dense subspaces closed under rational²⁴ linear combinations; this gives rise to an effective $\kappa + 1$ -Borel subset \mathfrak{B}_κ of a space of type κ , which can be regarded as the standard Borel κ -space of all Banach spaces of density κ .

Let B be a (real) Banach space of density κ with norm $\|\cdot\|_B$, and let $D = \{b_\alpha \mid \alpha < \kappa\}$ be a dense subset of B which is also closed under rational linear combinations. Without loss of generality, we can assume that b_0 is the zero vector of B (so that b_0 is the unique element of D with B -norm 0). Then we can identify B with an element $x_B = (x_B^+, x_B^\mathbb{Q}, x_B^{\|\cdot\|})$ of the space $\mathcal{X} := {}^{\kappa \times \kappa \times \kappa} 2 \times {}^{\kappa \times \mathbb{Q} \times \kappa} 2 \times {}^{\kappa \times \mathbb{Q}^+} 2$ by setting for $\alpha, \beta, \gamma < \kappa$, $p \in \mathbb{Q}$, and $q \in \mathbb{Q}^+$

$$(7.7) \quad \begin{aligned} x_B^+(\alpha, \beta, \gamma) &= 1 \Leftrightarrow b_\alpha + b_\beta = b_\gamma \\ x_B^\mathbb{Q}(\alpha, p, \beta) &= 1 \Leftrightarrow p \cdot b_\alpha = b_\beta \\ x_B^{\|\cdot\|}(\alpha, q) &= 1 \Leftrightarrow \|b_\alpha\|_B < q. \end{aligned}$$

The function x_B codes all the necessary informations to retrieve the normed vector space structure of D , and hence of the whole B . This suggests to consider the space $\mathfrak{B}_\kappa \subseteq \mathcal{X}$ consisting of those $x = (x^+, x^\mathbb{Q}, x^{\|\cdot\|}) \in \mathcal{X}$ satisfying the conditions in Table 1. The meaning of the conditions in this table is clear: given $x \in \mathfrak{B}_\kappa$, we can consider the normed \mathbb{Q} -vector space D_x on κ equipped with the (well-defined) operations $+_x$ and \cdot_x and the (well-defined) norm $\|\cdot\|_x$ obtained by setting for $\alpha, \beta < \kappa$ and $p \in \mathbb{Q}$

$$\begin{aligned} \alpha +_x \beta &= \gamma \Leftrightarrow x^+(\alpha, \beta, \gamma) = 1 \\ p \cdot_x \alpha &= \beta \Leftrightarrow x^\mathbb{Q}(\alpha, p, \beta) = 1 \\ \|\alpha\|_x &:= \inf\{q \in \mathbb{Q}^+ \mid x^{\|\cdot\|}(\alpha, q) = 1\}. \end{aligned}$$

The Banach space obtained by completing the norm $\|\cdot\|_x$ is denoted by B_x . Then for every Banach space B of density κ we get that its code $x_B \in \mathcal{X}$ defined in (7.7) belongs to \mathfrak{B}_κ (and is such that B and B_{x_B} are linearly isometric), and, conversely, for every $x \in \mathfrak{B}_\kappa$ the space B_x is a Banach space of density κ . Moreover, the explicit definition from Table 1 shows that \mathfrak{B}_κ is an effective $\kappa + 1$ -Borel subset of \mathcal{X} , and hence it inherits from it the topology τ_b and the corresponding (effective) $\kappa + 1$ -Borel structure, which turns it into a standard Borel κ -space.

²⁴To simplify the presentation, in this paper we focus on *real* Banach spaces. However, by replacing \mathbb{Q} with $\mathbb{Q} + i\mathbb{Q}$ one can extend our results to complex Banach spaces of density κ .

$\forall \alpha, \beta < \kappa \exists! \gamma < \kappa [x^+(\alpha, \beta, \gamma) = 1]$
$\forall \alpha < \kappa \forall p \in \mathbb{Q} \exists! \beta < \kappa [x^{\mathbb{Q}}(\alpha, p, \beta) = 1]$
$\forall \alpha, \beta, \gamma, \delta, \epsilon, \zeta < \kappa [x^+(\beta, \gamma, \delta) = 1 \wedge x^+(\alpha, \delta, \epsilon) = 1 \wedge x^+(\alpha, \beta, \zeta) = 1 \Rightarrow x^+(\zeta, \gamma, \epsilon) = 1]$
$\forall \alpha, \beta, \gamma < \kappa [x^+(\alpha, \beta, \gamma) = 1 \Leftrightarrow x^+(\beta, \alpha, \gamma) = 1]$
$\forall \alpha < \kappa [x^+(\alpha, 0, \alpha) = 1]$
$\forall \alpha < \kappa \exists \beta < \kappa [x^+(\alpha, \beta, 0) = 1]$
$\forall \alpha, \beta, \gamma < \kappa \forall p, p' \in \mathbb{Q} [x^{\mathbb{Q}}(\alpha, p, \beta) = 1 \wedge x^{\mathbb{Q}}(\beta, p', \gamma) = 1 \Rightarrow x^{\mathbb{Q}}(\alpha, pp', \gamma) = 1]$
$\forall \alpha < \kappa [x^{\mathbb{Q}}(\alpha, 1, \alpha) = 1]$
$\forall \alpha, \beta, \gamma, \delta, \epsilon, \zeta < \kappa \forall p \in \mathbb{Q} [x^+(\alpha, \beta, \gamma) = 1 \wedge x^{\mathbb{Q}}(\gamma, p, \delta) = 1 \wedge x^{\mathbb{Q}}(\alpha, p, \epsilon) = 1 \wedge x^{\mathbb{Q}}(\beta, p, \zeta) = 1 \Rightarrow x^+(\epsilon, \zeta, \delta) = 1]$
$\forall \alpha, \beta, \gamma, \delta < \kappa \forall p, p' \in \mathbb{Q}^+ [x^{\mathbb{Q}}(\alpha, p + p', \beta) = 1 \wedge x^{\mathbb{Q}}(\alpha, p, \gamma) = 1 \wedge x^{\mathbb{Q}}(\alpha, p', \delta) = 1 \Rightarrow x^+(\gamma, \delta, \beta) = 1]$
$\forall \alpha < \kappa \forall q, q' \in \mathbb{Q}^+ [q \leq q' \Rightarrow x^{\ \cdot\ }(\alpha, q) \leq x^{\ \cdot\ }(\alpha, q')]$
$\forall \alpha < \kappa \exists q \in \mathbb{Q}^+ [x^{\ \cdot\ }(\alpha, q) = 1]$
$\forall 0 < \alpha < \kappa \exists q \in \mathbb{Q}^+ [x^{\ \cdot\ }(\alpha, q) = 0]$
$\forall q \in \mathbb{Q}^+ [x^{\ \cdot\ }(0, q) = 1]$
$\forall \alpha, \beta < \kappa \forall p \in \mathbb{Q} \forall q \in \mathbb{Q}^+ [x^{\mathbb{Q}}(\alpha, p, \beta) = 1 \Rightarrow (x^{\ \cdot\ }(\beta, q) = 1 \Leftrightarrow x^{\ \cdot\ }(\alpha, q/ p) = 1)]$
$\forall \alpha, \beta, \gamma < \kappa \forall q, q' \in \mathbb{Q}^+ [x^+(\alpha, \beta, \gamma) = 1 \wedge x^{\ \cdot\ }(\alpha, q) = 1 \wedge x^{\ \cdot\ }(\beta, q') = 1 \Rightarrow x^{\ \cdot\ }(\gamma, q + q') = 1]$
$\forall \alpha < \kappa \exists \beta < \kappa \exists q \in \mathbb{Q}^+ \forall \gamma < \alpha \forall \delta, \epsilon < \kappa [x^{\mathbb{Q}}(\gamma, -1, \delta) = 1 \wedge x^+(\beta, \delta, \epsilon) = 1 \Rightarrow x^{\ \cdot\ }(\epsilon, q) = 0]$

TABLE 1. The conditions defining \mathfrak{B}_κ .

Thus we can regard \mathfrak{B}_κ as the **space of** (codes for) **all Banach spaces of density** κ (up to linear isometry).

The **linear isometric embeddability** relation \sqsubseteq^{li} on \mathfrak{B}_κ is given by

$$x \sqsubseteq^{li} y \Leftrightarrow \text{there is a linear norm-preserving map from } B_x \text{ into } B_y,$$

while the **linear isometry** relation \cong^{li} on \mathfrak{B}_κ is given by

$$x \cong^{li} y \Leftrightarrow \text{there is a linear norm-preserving bijection between } B_x \text{ and } B_y.$$

For every $x, y \in \mathfrak{B}_\kappa$ one has that $x \sqsubseteq^{li} y$ if and only if there is a linear norm-preserving map $i: D_x \rightarrow B_y$. Using the Tarski-Kuratowski algorithm, the relation \sqsubseteq^{li} is a κ -analytic quasi-order on \mathfrak{B}_κ , and \cong^{li} is a κ -analytic equivalence relation on \mathfrak{B}_κ .

8. Infinitary logics and models

The topic of this section is infinitary logics (as presented e.g. in [Bar69]) and models of infinitary sentences. As in Section 7.2.2, $\mathcal{L} = \{R_i \mid i \in I\}$ with $I \in \omega$ denotes a finite relational signature, and n_i is the arity of the symbol R_i . As usual λ, κ denote infinite cardinals.

8.1. Infinitary logics.

8.1.1. Syntax.

DEFINITION 8.1. For $\lambda \leq \kappa$ the set $\mathcal{L}_{\kappa\lambda}$ of infinitary formulæ for the signature \mathcal{L} is defined as follows.

- Fix a list of objects $\langle v_\alpha \mid \alpha < \lambda \rangle$ called **variables**.

- The **atomic formulæ** are finite sequences of the form $\langle \simeq, v_{\alpha_1}, v_{\alpha_2} \rangle$ and $\langle R_i, v_{\alpha_1}, \dots, v_{\alpha_{n_i}} \rangle$, with $i < I$ and $\alpha_1, \dots, \alpha_{n_i} < \lambda$.²⁵
- $\mathcal{L}_{\kappa\lambda}$ is the smallest collection containing the atomic formulæ and closed under the following operations:
 - negation:** $\varphi \mapsto \langle \neg \rangle \frown \varphi$;
 - generalized conjunctions:** if $\varphi_\alpha \in \mathcal{L}_{\kappa\lambda}$ (for $\alpha < \nu < \kappa$), and if the total number of variables that occur free in some φ_α is $< \lambda$, then $\langle \bigwedge \rangle \frown \langle \varphi_\alpha \mid \alpha < \nu \rangle \in \mathcal{L}_{\kappa\lambda}$;
 - generalized existential quantification:** $\varphi \mapsto \langle \exists \rangle \frown \langle v_{u(\alpha)} \mid \alpha < \nu \rangle \frown \varphi$, for an increasing $u: \nu \rightarrow \lambda$ and $\nu < \lambda$.

REMARKS 8.2. (i) Each formula in $\mathcal{L}_{\kappa\lambda}$ has $< \lambda$ free variables occurring in it. In fact the formal definition of $\varphi \in \mathcal{L}_{\kappa\lambda}$ requires the simultaneous definition of

$$\text{Fv}(\varphi) \in [\lambda]^{<\lambda},$$

the set of all $\alpha \in \lambda$ such that v_α occurs free in φ .

- (ii) Formally, the generalized conjunction of the formulæ $\varphi_\alpha \in \mathcal{L}_{\kappa\lambda}$ (for $\alpha < \nu < \kappa$) should be defined as the concatenation $\langle \bigwedge \rangle \frown \varphi_0 \frown \varphi_1 \frown \dots \frown \varphi_\alpha \frown \dots$ rather than $\langle \bigwedge \rangle \frown \langle \varphi_\alpha \mid \alpha < \nu \rangle$, so that each formula in $\mathcal{L}_{\kappa\lambda}$ is always a(n infinite) sequence of logical symbols and symbols from \mathcal{L} . However, this other approach would then require us to prove a unique readability lemma to ensure that each of the formulæ φ_α can be recovered from their conjunction, something which is clear with our current definition. This formal presentation would be considerably more opaque, so we decided to abandon it in favor of clarity.
- (iii) $\mathcal{L}_{\omega\omega}$ is ordinary first-order logic.

The usual syntactical notions (subformula, sentence, etc.) are defined in the obvious way, and each φ is completely described by the tree of its subformulæ $\text{Subf}(\varphi)$, which is called the **syntactical tree** of φ .

It is convenient to assume that all symbols of $\mathcal{L}_{\kappa\lambda}$ are construed as fixed elements of λ — for example we can encode v_α by the ordinal $\langle 0, \alpha \rangle$, the equality predicate \simeq by $\langle 1, 0 \rangle$, the negation \neg by $\langle 1, 1 \rangle$, the generalized conjunction \bigwedge by $\langle 1, 2 \rangle$, and R_i by $\langle i + 2, n_i \rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing function of (2.1). In this way, any atomic formula can be encoded by a finite sequence of ordinals $< \lambda$ (and therefore by an ordinal $< \lambda$ by (2.2)), and an arbitrary $\varphi \in \mathcal{L}_{\kappa\lambda}$ can be encoded via $\text{Subf}(\varphi)$ with a tree on κ . In fact: $\text{Subf}(\varphi)$ can be construed as a *labelled* $< \kappa$ -branching descriptive set-theoretic tree on κ of height $\leq \omega$, with labels in $[\lambda]^{<\lambda}$. Notice that also formulæ in $\mathcal{L}_{\kappa+\lambda}$ may be encoded through their syntactical tree as (labelled) tree on κ — the difference with the previous case is that now we have to consider κ -branching trees. Thus:

- the set $\mathcal{L}_{\kappa\lambda}$ can be defined in every transitive model of ZF containing κ ,
- the predicate “ $\varphi \in \mathcal{L}_{\kappa\lambda}$ ” is absolute for such models,
- $\mathcal{L}_{\kappa'\lambda'} \subseteq \mathcal{L}_{\kappa\lambda}$ whenever $\kappa' \leq \kappa$ and $\lambda' \leq \lambda$.

We forsake the official, but awkward, notation for a language in favor of a more relaxed (if a bit inaccurate) one. Thus:

- the atomic formulæ are written as $v_{\alpha_1} \simeq v_{\alpha_2}$ and $R_i(v_{\alpha_1}, \dots, v_{\alpha_{n_i}})$, and $v_{\alpha_1} \neq v_{\alpha_2}$ is the negation of $v_{\alpha_1} \simeq v_{\alpha_2}$;
- the letters φ, ψ range over formulæ, while σ ranges over sentences;
- the negation, the generalized conjunction, and the generalized existential quantification are written as $\neg\varphi$, $\bigwedge_{\alpha < \nu} \varphi_\alpha$, and $\exists \langle v_{u(\alpha)} \mid \alpha < \nu \rangle \varphi$ or $\exists v_{u(0)} \exists v_{u(1)} \dots \varphi$ or²⁶ $\exists v_u \varphi$ (with $u \in$

²⁵We use the symbol \simeq for the equality predicate in the infinitary logic, to distinguish it from the usual $=$ used in the language of set theory.

²⁶Recall from Section 2.1.2 that we are identifying each $u \in [\lambda]^{<\lambda}$ with its increasing enumerating function.

$[\lambda]^{<\lambda}$). The generalized disjunction $\bigvee_{\alpha < \nu} \varphi_\alpha$ and generalized universal quantification $\forall \langle v_{u(\alpha)} \mid \alpha < \nu \rangle \varphi$ or $\forall v_{u(0)} \forall v_{u(1)} \cdots \varphi$ or $\forall v_u \varphi$ are defined by means of the de Morgan's laws. Ordinary conjunctions and disjunctions are obtained from generalized ones by setting $\nu = 2$, and from these all other connectives are defined. Similarly, ordinary quantifications are obtained from generalized ones by setting $\nu = 1$;

- the letters x, y, z, \dots range over $\{v_\alpha \mid \alpha < \lambda\}$, so that $\exists \langle x_\alpha \mid \alpha < \nu \rangle \varphi$ is $\exists \langle v_{u(\alpha)} \mid \alpha < \nu \rangle \varphi$ for some $u: \nu \rightarrow \lambda$. Since cardinals are additively closed, $\exists \langle x_\alpha \mid \alpha < \nu \rangle \exists \langle y_\beta \mid \beta < \xi \rangle \varphi$ is identified with $\exists \langle x_\alpha \mid \alpha < \nu \rangle \wedge \langle y_\beta \mid \beta < \xi \rangle \varphi$,
- the expression $\varphi(\langle x_\alpha \mid \alpha < \nu \rangle)$ means that the variables that occur free in φ are among the $\{x_\alpha \mid \alpha < \nu\}$, and we will also assume that the x_α are distinct and listed in an increasing order with respect to the official list. Therefore $\langle x_\alpha \mid \alpha < \nu \rangle = \langle v_{u(\alpha)} \mid \alpha < \nu \rangle$ for some unique increasing $u: \nu \rightarrow \lambda$. Such u can be identified with its range, which is an element of $[\lambda]^\nu$, so we write $\varphi(v_u)$ with $u \in [\lambda]^\nu$ to say that if v_α occurs free in φ , then $\alpha \in u$.

DEFINITION 8.3. (i) The set $\mathcal{L}_{\kappa\lambda}^0$ of all **propositional formulæ** consists of those $\varphi \in \mathcal{L}_{\kappa\lambda}$ obtained from the atomic formulæ using only negation and generalized conjunctions.

(ii) The set $\mathcal{L}_{\kappa\lambda}^b$ of all **bounded formulæ** consists of those $\varphi \in \mathcal{L}_{\kappa\lambda}$ such that:

if $\exists \langle x_\alpha \mid \alpha < \nu \rangle \psi \in \text{Subf}(\varphi)$ and $\nu \geq \omega$, then $\psi \in \mathcal{L}_{\kappa'\lambda}^0$ for some $\kappa' < \kappa$.

Thus $\mathcal{L}_{\kappa\lambda}^b$ is closed under negation, infinitary conjunctions and disjunctions (of size $< \kappa$), finite quantifications, but *not* under infinitary quantifications. The reason for investigating this technical notion is that $\mathcal{L}_{\kappa\lambda}^b$ avoids the counterexamples related to (the relevant direction of) the generalized Lopez-Escobar theorem (see Remark 8.12). This feature of $\mathcal{L}_{\kappa\lambda}^b$ is crucial for the results of Section 14.1 (see Theorems 14.8 and 14.10).

REMARKS 8.4. (i) We are mostly interested in logics of the form $\mathcal{L}_{\kappa^+\kappa}$, $\mathcal{L}_{\kappa^+\kappa}^b$ and $\mathcal{L}_{\kappa^+\lambda}$ for $\lambda < \kappa$.

(ii) As for α -Borel sets, one could define the logics $\mathcal{L}_{\alpha\beta}$ for arbitrary $\beta \leq \alpha \in \text{Ord}$ in the obvious way, and with such definition one could write $\mathcal{L}_{(\kappa+1)\kappa}$, $\mathcal{L}_{(\kappa+1)\kappa}^b$ and $\mathcal{L}_{(\kappa+1)\lambda}$ instead of $\mathcal{L}_{\kappa^+\kappa}$, $\mathcal{L}_{\kappa^+\kappa}^b$ and $\mathcal{L}_{\kappa^+\lambda}$. In principle, this would be preferable, as $\kappa + 1$ is absolute while κ^+ is not. However, there are two reasons to eschew such move. Firstly, this would run against standard notation in the literature. Secondly, this could be source of endless minor notational quibbles regarding the use of variables in the generalized existential quantification.

(iii) A more substantial move would be to extend the definition of $\mathcal{L}_{\kappa\lambda}$ to $\mathcal{L}_{A+1 B+1}$ with A, B arbitrary sets, just like \mathbf{B}_{J+1} is a generalization of \mathbf{B}_α (Definition 4.8). This would allow us to give a ZF-formulation of certain results (Proposition 8.13 parts (b) and (c), and Proposition 8.14). On the other hand, besides the technical problems already mentioned for $\mathcal{L}_{\alpha\beta}$, this would render the notation quite opaque. As we have no use for $\mathcal{L}_{A+1 B+1}$, we decided to abandon them altogether.

8.1.2. *Semantics.* If $\mathcal{A} = \langle A; R_i^A \rangle_{i \in I}$ is an \mathcal{L} -structure, $\varphi(v_u) \in \mathcal{L}_{\kappa\lambda}$ with $u \in [\lambda]^{<\lambda}$, and $s: u \rightarrow A$, then

$$\mathcal{A} \models \varphi[s]$$

means that the formula obtained from $\varphi(v_u)$ by substituting each v_α with $s(\alpha)$ for all $\alpha \in u$, holds true in the structure \mathcal{A} . This notion is defined recursively on the tree $\text{Subf}(\varphi)$ of all subformulæ of φ . For example, if φ is $v_\alpha \simeq v_\beta$ or $R_i(v_{\alpha_1}, \dots, v_{\alpha_n})$, then $\mathcal{A} \models \varphi[s]$ if and only if $s(\alpha) = s(\beta)$ or $\langle s(\alpha_1), \dots, s(\alpha_n) \rangle \in R_i^A$. If instead $\exists v_u \psi$, then letting $w := u \cup \text{Fv}(\psi)$

$$(8.1) \quad \mathcal{A} \models \varphi[s] \Leftrightarrow \exists t \in {}^w A \ (t \restriction \text{Fv}(\varphi) = s \restriction \text{Fv}(\varphi) \wedge \mathcal{A} \models \psi[t]).$$

As in the first-order case, if $s \restriction \text{Fv}(\varphi) = t \restriction \text{Fv}(\varphi)$ then $\mathcal{A} \models \varphi[s] \Leftrightarrow \mathcal{A} \models \varphi[t]$, so if φ is a sentence, i.e. $\text{Fv}(\varphi) = \emptyset$, then the truth $\mathcal{A} \models \varphi[s]$ does not depend on s and we write $\mathcal{A} \models \varphi$.

REMARKS 8.5. (i) First-order formulæ can be identified with specific natural numbers and $\mathcal{L}_{\omega\omega}$ can be identified with a subset of ω . In particular, both $\mathcal{L}_{\omega\omega}$ and its members are well-orderable. This fails badly for $\mathcal{L}_{\kappa\lambda}$ when $\kappa > \omega$. In fact, while atomic formulæ (and their negations) can be coded as ordinals $< \lambda$, by taking countable conjunctions it is possible to inject ${}^\omega 2$ into $\mathcal{L}_{\kappa\lambda}^0$.

- (ii) By (8.1), when $\lambda \geq \omega_1$ the existence of Skolem functions for \mathcal{A} requires AC (or at least a well-ordering of ${}^{<\lambda}\lambda$), even when \mathcal{A} is well-orderable.
- (iii) The satisfaction relation for $\mathcal{L}_{\kappa\lambda}$ with $\kappa \geq \omega_2$ and $\lambda \geq \omega_1$ is not absolute for transitive models of ZFC. In fact there can be a countable structure \mathcal{A} with domain ω and an $\mathcal{L}_{\omega_2\omega_1}$ -sentence σ such that $\mathcal{A} \models \sigma$ in the universe V , but $\mathcal{A} \not\models \sigma$ in a suitable forcing extension $V[G]$ of V . To see this, assume that $\text{ZFC} + \text{CH}$ holds in V , let $\mathcal{L} = \{P\}$ be a language consisting of just one unary relational symbol, and let $\mathcal{A} = \langle \omega; P^{\mathcal{A}} \rangle \in V$ be such that $P^{\mathcal{A}}$ is infinite and coinfinite. For the sake of definiteness set $P^{\mathcal{A}}(i) \Leftrightarrow i$ is even (for $i \in \omega$), so that \mathcal{A} belongs to any model of ZF. Let

$$\mathcal{D} := \{D \in V \mid V \models \text{"}D \text{ is } \mathbf{Fn}(\omega, 2; \omega)\text{-dense"}\},$$

so that \mathcal{D} has size ω_1 in V by CH. The plan is to define an $\mathcal{L}_{\omega_2\omega_1}$ -sentence σ coding the existence of a \mathcal{D} -generic for the Cohen forcing $\mathbf{Fn}(\omega, 2; \omega)$. Given $s \in {}^{<\omega}2$, let $\psi_s = \psi_s(v_0, \dots, v_{\text{lh}(s)-1})$ be the $\mathcal{L}_{\omega\omega}$ -formula

$$\bigwedge_{\substack{i < \text{lh } s \\ s(i)=1}} P(v_i) \wedge \bigwedge_{\substack{i < \text{lh } s \\ s(i)=0}} \neg P(v_i).$$

Finally, let σ be the $\mathcal{L}_{\omega_2\omega_1}$ -sentence $\exists \langle v_i \mid i \in \omega \rangle \varphi$, where $\varphi = \varphi(\langle v_i \mid i \in \omega \rangle)$ is the formula

$$\left(\bigwedge_{i < j < \omega} v_i \neq v_j \right) \wedge \bigwedge_{D \in \mathcal{D}} \bigvee_{s \in D} \psi_s(v_0, \dots, v_{\text{lh}(s)-1}).$$

It is not hard to check that working in any ZF-model $W \supseteq V$, if $\langle a_i \mid i \in \omega \rangle$ is a sequence of elements of \mathcal{A} such that $\mathcal{A} \models \varphi(\langle a_i \mid i \in \omega \rangle)$, then the function $G: \omega \rightarrow 2$ defined by $G(i) = 1 \Leftrightarrow P^{\mathcal{A}}(a_i)$ is \mathcal{D} -generic for $\mathbf{Fn}(\omega, 2; \omega)$. Conversely, if $G: \omega \rightarrow 2$ is \mathcal{D} -generic for $\mathbf{Fn}(\omega, 2; \omega)$ then, using the fact that $\{i \in \omega \mid P^{\mathcal{A}}(i)\}$ is infinite and coinfinite, one can find a sequence $\langle a_i \mid i \in \omega \rangle$ of elements of \mathcal{A} such that $\mathcal{A} \models \varphi(\langle a_i \mid i \in \omega \rangle)$. Therefore, in any W as above it holds

$$\mathcal{A} \models \sigma \Leftrightarrow \exists G: \omega \rightarrow 2 (G \text{ is } \mathcal{D}\text{-generic for } \mathbf{Fn}(\omega, 2; \omega)).$$

Therefore $\mathcal{A} \not\models \sigma$ in V , but $\mathcal{A} \models \sigma$ in any $\mathbf{Fn}(\omega, 2; \omega)$ -generic extension of V .

As seen in Section 7.2.2, any \mathcal{L} -structure of size κ can be identified with an element of $\text{Mod}_{\mathcal{L}}^{\kappa}$ (which is a typical example of a space of type κ). For $\varphi(v_u) \in \mathcal{L}_{\nu\mu}$ and $u \in [\mu]^{<\mu}$ we set

$$(8.2) \quad M_{\varphi, u} := \{(X, s) \in \text{Mod}_{\mathcal{L}}^{\kappa} \times {}^u \kappa \mid X \models \varphi[s]\},$$

and if σ is a sentence, we set

$$(8.3) \quad \text{Mod}_{\sigma}^{\kappa} := \{X \in \text{Mod}_{\mathcal{L}}^{\kappa} \mid X \models \sigma\}.$$

Thus $\text{Mod}_{\sigma}^{\kappa}$ is the space of all \mathcal{L} -structures with domain κ which satisfy σ . We also let

$$\text{Mod}_{\sigma}^{<\kappa} := \bigcup_{\lambda < \kappa} \text{Mod}_{\sigma}^{\lambda} \quad \text{and} \quad \text{Mod}_{\sigma}^{\infty} := \bigcup_{\kappa \in \text{Card}} \text{Mod}_{\sigma}^{\kappa}.$$

Notice that sets of the form $\text{Mod}_{\sigma}^{\kappa}$, $\text{Mod}_{\sigma}^{<\kappa}$, and $\text{Mod}_{\sigma}^{\infty}$ are always invariant under isomorphism.

NOTATION. From now on we use \sqsubset_σ^κ and \cong_σ^κ for the embeddability and isomorphism relations restricted to the space of models Mod_σ^κ . Similarly, we denote the restriction of the embeddability (respectively, isomorphism) relation to the spaces $\text{Mod}_\sigma^{<\kappa}$ and Mod_σ^∞ by $\sqsubset_\sigma^{<\kappa}$ and \sqsubset_σ^∞ (respectively, $\cong_\sigma^{<\kappa}$ and \cong_σ^∞). A similar notation is adopted for the less frequently used bi-embeddability relation \approx as well: in this case we write \approx_σ^κ , $\approx_\sigma^{<\kappa}$, and \approx_σ^∞ to denote the restriction of \approx to, respectively, Mod_σ^κ , $\text{Mod}_\sigma^{<\kappa}$, and Mod_σ^∞ .

REMARK 8.6. Fix any $\sigma \in \mathcal{L}_{\kappa+\kappa}$. As we shall see in the next section, when $\kappa < \kappa^{<\kappa}$ (or when the assumption AC is dropped) the set Mod_σ^κ may fail to be a $\kappa+1$ -Borel subset of Mod_σ^κ , even when considering the finest topology τ_b , see Remark 8.12. However, it is worth to point out that if κ is regular and either $\sigma \in \mathcal{L}_{\kappa+\kappa}^b$ or $\text{AC} + \kappa^{<\kappa} = \kappa$ holds, then $\text{Mod}_\sigma^\kappa \in \mathbf{B}_{\kappa+1}^e(\text{Mod}_\sigma^\kappa, \tau_b)$ by Corollary 8.10(b) and Theorem 8.7. In this case Mod_σ^κ is a standard Borel κ -space and the relations \sqsubset_σ^κ , \cong_σ^κ , and \approx_σ^κ are, respectively, a κ -analytic quasi-order and two κ -analytic equivalence relations.

The topological complexity of Mod_σ^κ as a subspace of the space Mod_σ^κ of type κ (with respect to the various natural topologies on it, see Definition 7.9) is the subject of Section 8.2. Towards this goal, we equip the spaces $\text{Mod}_\sigma^\kappa \times {}^u\kappa$ in (8.2) with the product of the topology τ on Mod_σ^κ and σ on ${}^u\kappa$, where τ is either the product topology, the λ -topology ($\omega \leq \lambda < \max(\text{cof}(\kappa)^+, \kappa)$), or the bounded topology, and σ is the discrete topology on ${}^u\kappa$. As before, the resulting topologies are called, respectively, product topology, λ -topology, and bounded topology, and are denoted by τ_p , τ_λ , and τ_b . The bijection

$$(8.4) \quad \text{Mod}_\sigma^\kappa \times {}^\emptyset\kappa \rightarrow \text{Mod}_\sigma^\kappa, \quad (X, \emptyset) \mapsto X$$

is a homeomorphism (when on both sides Mod_σ^κ is endowed with the same topology) witnessing the identification between $M_{\sigma, \emptyset}$ and Mod_σ^κ for every $\sigma \in \mathcal{L}_{\kappa\lambda}$.

8.2. Some generalizations of the Lopez-Escobar theorem. A theorem of Lopez-Escobar (see e.g. [Kec95, Theorem 16.8]) says that $A \subseteq \text{Mod}_\sigma^\omega$ is Borel and invariant under isomorphism if and only if $A = \text{Mod}_\sigma^\omega$ for some $\sigma \in \mathcal{L}_{\omega_1\omega}$. The aim of this section is to extend this to an arbitrary infinite cardinal κ . Full generalizations of the Lopez-Escobar theorem have been studied when $\kappa = \omega_1$ and $\text{AC} + \text{CH}$ holds, see e.g. [Vau75]. The next result is a further generalization to cardinals $\kappa = \kappa^{<\kappa}$, and has been independently obtained in [FHK14]. It follows immediately from Corollary 8.11 with $\lambda = \kappa$, and from Proposition 8.14(c) below.

THEOREM 8.7 (AC). *If $\kappa^{<\kappa} = \kappa$, then a set $A \subseteq \text{Mod}_\sigma^\kappa$ is $\kappa+1$ -Borel (with respect to τ_b) and closed under isomorphism if and only if $A = \text{Mod}_\sigma^\kappa$ for some $\sigma \in \mathcal{L}_{\kappa+\kappa}$.*

Since $\mathbf{B}_{\kappa+1}(\tau_p) \subseteq \mathbf{B}_{\kappa+1}(\tau_\lambda) \subseteq \mathbf{B}_{\kappa+1}(\tau_b)$, Theorem 8.7 concerns the largest possible class of $\kappa+1$ -Borel sets, but in fact under $\kappa^{<\kappa} = \kappa$ all the above classes coincide by Corollary 4.10. Remarks 8.12 and 8.17 show that without the assumption $\kappa^{<\kappa} = \kappa$ both directions of the equivalence in Theorem 8.7 may fail. The assumption $\kappa^{<\kappa} = \kappa$ in Theorem 8.7 is inconvenient for our work because, as already explained in the introduction:

- we need to apply a generalization of the Lopez-Escobar theorem in models where AC fails (e.g. in models of determinacy);
- even when working in models satisfying AC, our results concern uncountable cardinals $\kappa < |\omega 2|$, which cannot satisfy $\kappa^{<\kappa} = \kappa$.

However, a careful analysis of the proof of Theorem 8.7 reveals some new intermediate results that may be useful. In one direction, Corollaries 8.10 and 8.11 show that if the sentence σ is chosen in a suitable fragment of $\mathcal{L}_{\kappa+\kappa}$, then Mod_σ^κ may turn out to be $\kappa+1$ -Borel even if the assumptions AC and $\kappa^{<\kappa} = \kappa$ are dropped — in fact, the bounded version $\mathcal{L}_{\kappa+\kappa}^b$ of the logic

$\mathcal{L}_{\kappa+\kappa}$ has been introduced for this purpose. In another direction, Proposition 8.14 gives some interesting results even in situations when Theorem 8.7 cannot be applied. For example, part (a) yields in ZFC that if $\text{Mod}_{\mathcal{L}}^{\omega_1}$ is endowed with the product topology, then every $\omega_1 + 1$ -Borel set $A \subseteq \text{Mod}_{\mathcal{L}}^{\omega_1}$ which is invariant under isomorphism is of the form $\text{Mod}_{\sigma}^{\omega_1}$ for some $\sigma \in \mathcal{L}_{(2^{\aleph_0})+\omega_1}$, independently of CH. More generally, when $\kappa = \mu^+$ and $\text{Mod}_{\mathcal{L}}^{\kappa}$ is endowed with the product topology, then every $\kappa + 1$ -Borel set $A \subseteq \text{Mod}_{\mathcal{L}}^{\kappa}$ invariant under isomorphism is always of the form $\text{Mod}_{\sigma}^{\kappa}$ for some $\mathcal{L}_{(2^{\mu})+\kappa}$ -sentence σ . Similarly, if $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = \aleph_3$ then Theorem 8.7 cannot be applied with $\kappa = \omega_2$. However Proposition 8.14(b) gives that if $\text{Mod}_{\mathcal{L}}^{\omega_2}$ is endowed with the ω_1 -topology, then every $\omega_2 + 1$ -Borel set $A \subseteq \text{Mod}_{\mathcal{L}}^{\omega_2}$ invariant under isomorphism is of the form $\text{Mod}_{\sigma}^{\omega_2}$ for some $\sigma \in \mathcal{L}_{\omega_4\omega_2}$.

8.2.1. *From formulæ to invariant Borel sets.* In the next results we use the sets $M_{\varphi,u}$ defined in (8.2). The proof of the following lemma is a straightforward adaptation of the proof of [Kec95, Proposition 16.17].

LEMMA 8.8. *Let $\lambda \leq \kappa$ and $\varphi(v_u) \in \mathcal{L}_{\kappa\lambda}$ with $u \in [\lambda]^{<\lambda}$. The following hold:*

- (a) *If φ is atomic then $M_{\varphi,u}$ is a basic clopen set with respect to τ_p (and hence also with respect to τ_{λ} and τ_b).*
- (b) *If $\varphi = \neg\psi$ then $M_{\varphi,u} = (\text{Mod}_{\mathcal{L}}^{\kappa} \times {}^u\kappa) \setminus M_{\psi,u}$.*
- (c) *If $\varphi = \bigwedge_{\alpha < \nu} \psi_{\alpha}$, where $\nu < \kappa$, then $M_{\varphi,u} = \bigcap_{\alpha < \nu} M_{\psi_{\alpha},u}$.*
- (d) *If $\varphi = \exists v_u \psi$ for some $u \in [\lambda]^{<\lambda}$, let $w := u \cup \text{Fv}(\psi)$. Then*

$$M_{\varphi,u} = \bigcup_{t \in {}^w\kappa} \pi_t^{-1}(M_{\psi,w}),$$

where $\pi_t: \text{Mod}_{\mathcal{L}}^{\kappa} \times {}^u\kappa \rightarrow \text{Mod}_{\mathcal{L}}^{\kappa} \times {}^w\kappa$ is the continuous function

$$(x, s) \mapsto (x, s \upharpoonright \text{Fv}(\varphi) \cup t \upharpoonright (w \setminus \text{Fv}(\varphi))).$$

In the following results of this section, the ambient space, unless otherwise indicated, is $\text{Mod}_{\mathcal{L}}^{\kappa}$. Recall from Definition 4.8 that for J an arbitrary set and \mathcal{B}_* the canonical basis for the topology $\tau_* \in \{\tau_p, \tau_{\lambda}, \tau_b\}$, we denote by $\mathbf{B}_{J+1}^e(\mathcal{B}_*)$ the collection of all effective $J + 1$ -Borel subsets of $\text{Mod}_{\mathcal{L}}^{\kappa}$ with codes taking value in \mathcal{B}_* .

PROPOSITION 8.9. *Let $\lambda \leq \kappa$ and $\varphi(v_u) \in \mathcal{L}_{\kappa+\lambda}$ with $u \in [\lambda]^{<\lambda}$. Then the following hold:*

- (a) *$M_{\varphi,u} \in \mathbf{B}_{J+1}^e(\tau_p) \subseteq \mathbf{B}_{J+1}^e(\tau_{\lambda}) \subseteq \mathbf{B}_{J+1}^e(\tau_b)$, where $J := {}^{<\lambda}\kappa$.*
- (b) *If κ is regular and $\varphi \in \mathcal{L}_{\kappa\lambda}^0$, then $M_{\varphi,u}$ is τ_b -clopen.*
- (c) *If κ is regular and $\varphi \in \mathcal{L}_{\kappa+\lambda}^b$, then $M_{\varphi,u} \in \mathbf{B}_{\kappa+1}^e(\tau_b)$.*
- (d) *If $\lambda = \omega$, then $M_{\varphi,u} \in \mathbf{B}_{\kappa+1}^e(\tau_p)$.*

PROOF. For part (a) it is enough to argue by induction on the complexity of the subformulæ of φ and use Lemma 8.8 to construct an effective ${}^{<\lambda}\kappa + 1$ -Borel code (T, ϕ) for $M_{\varphi,u}$. We inductively define a function f from the set of the subformulæ ψ of φ to the set of effective ${}^{<\lambda}\kappa + 1$ -Borel codes such that $f(\psi)$ gives a code for $M_{\psi,u}$. (The function f allows to avoid the use of AC in the inductive steps.) The tree T is obtained from $\text{Subf}(\varphi)$, the syntactical tree of the subformulæ of φ , by replacing every non-splitting node given by an infinitary existential quantifier $\exists \langle x_{\alpha} \mid \alpha < \mu \rangle$ with a splitting node of T with (at most) ${}^{\mu}\kappa$ -many immediate successors, each corresponding to a possible witness of the existential quantification.

For part (b) argue as follows. By inductively applying Lemma 8.8 to the subformulæ of φ as in part (a), we obtain that $M_{\varphi,u} \in \text{Alg}(\mathcal{C}, \kappa)$ where \mathcal{C} is the collection of all basic τ_p -clopen subsets of $\text{Mod}_{\mathcal{L}}^{\kappa} \times {}^u\kappa$. Each of these sets is thus τ_b -clopen by equation (3.1) on page 20 (see the comments following Definition 7.9).

The proof of part (c) is similar to that of part (a). In constructing an effective $\kappa + 1$ -Borel code for $M_{\varphi, u}$, unions of size $> \kappa$ are taken only when an infinitary existential quantification is encountered. By the way $\mathcal{L}_{\kappa+\lambda}^b$ is defined (see Definition 8.3), the quantified subformula must be in $\mathcal{L}_{\kappa\lambda}^0$, and hence such a union is in τ_b by part (b). The result then follows by inductively applying Lemma 8.8 again.

Part (d) follows from part (a) and the fact that $<^\omega \kappa$ is in bijection with κ . \square

By (8.4), setting $u = \emptyset$ in Proposition 8.9 we get:

COROLLARY 8.10. *Suppose $\lambda \leq \kappa$.*

- (a) *If $\sigma \in \mathcal{L}_{\kappa+\lambda}$, then $\text{Mod}_\sigma^\kappa \in \mathbf{B}_{J+1}^e(\tau_p) \subseteq \mathbf{B}_{J+1}^e(\tau_b)$, where $J := <^\lambda \kappa$.*
- (b) *If κ is regular and $\sigma \in \mathcal{L}_{\kappa+\kappa}^b$, then $\text{Mod}_\sigma^\kappa \in \mathbf{B}_{\kappa+1}^e(\tau_b)$.*
- (c) *If $\sigma \in \mathcal{L}_{\kappa+\omega}$, then $\text{Mod}_\sigma^\kappa \in \mathbf{B}_{\kappa+1}^e(\tau_p)$.*

Using Proposition 8.9(a) and Corollary 8.10(a) we also have:

COROLLARY 8.11 (AC). *Suppose $\lambda \leq \kappa$ and $\kappa^{<\lambda} = \kappa$, and let $\varphi(v_u) \in \mathcal{L}_{\kappa+\lambda}$ with $u \in [\lambda]^{<\lambda}$. Then $M_{\varphi, u} \in \mathbf{B}_{\kappa+1}^e(\tau_p) \subseteq \mathbf{B}_{\kappa+1}^e(\tau_b)$.*

Similarly, if $\sigma \in \mathcal{L}_{\kappa+\lambda}$ is a sentence, then $\text{Mod}_\sigma^\kappa \in \mathbf{B}_{\kappa+1}^e(\tau_p) \subseteq \mathbf{B}_{\kappa+1}^e(\tau_b)$.

REMARK 8.12. If AC holds and $\kappa^{<\kappa} \neq \kappa$, then there may be sentences $\sigma \in \mathcal{L}_{\kappa+\kappa}$ such that Mod_σ^κ is not $\kappa + 1$ -Borel with respect to the finest topology considered here, namely the bounded topology τ_b . As observed in [FHK14, Remark 25], by work of Väänänen and Shelah, if $\lambda^+ = \kappa < \kappa^{<\kappa}$ and $\lambda^{<\lambda} = \lambda$ and a forcing axiom holds (and $\omega_1^L = \omega_1$ if $\lambda = \omega$), then for some $\sigma \in \mathcal{L}_{\kappa\kappa} \subseteq \mathcal{L}_{\kappa+\kappa}$ the set Mod_σ^κ does not have the κ -Baire property, so it is not $\kappa + 1$ -Borel with respect to τ_b by Proposition 6.14. Corollary 8.10(b) is thus one of the main reasons to introduce the bounded logic $\mathcal{L}_{\kappa+\kappa}^b$.

8.2.2. *From invariant Borel sets to formulae.* Let

$$\text{Inv}_{\mathcal{L}}^\kappa := \{A \subseteq \text{Mod}_{\mathcal{L}}^\kappa \mid \forall X, Y \in \text{Mod}_{\mathcal{L}}^\kappa (X \in A \wedge Y \cong X \Rightarrow Y \in A)\}$$

be the family of all subsets of $\text{Mod}_{\mathcal{L}}^\kappa$ which are **invariant under isomorphism**. For every $\sigma \in \mathcal{L}_{\nu\mu}$ and $\mu \leq \nu'$ infinite cardinals, $\text{Mod}_\sigma^\kappa \in \text{Inv}_{\mathcal{L}}^\kappa$. The following propositions provide a partial converse to Corollaries 8.10 and 8.11.

PROPOSITION 8.13. *Suppose κ, λ are infinite cardinals with $\omega \leq \lambda < \max(\text{cof}(\kappa)^+, \kappa)$.*

- (a) $\mathbf{B}_{\omega+1}^e(\mathcal{B}_p) \cap \text{Inv}_{\mathcal{L}}^\kappa \subseteq \{\text{Mod}_\sigma^\kappa \mid \sigma \in \mathcal{L}_{\kappa+\omega}\}$.
- (b) *Assume AC. Then $\mathbf{B}_{\text{cof}(\lambda)+1}^e(\mathcal{B}_\lambda) \cap \text{Inv}_{\mathcal{L}}^\kappa \subseteq \{\text{Mod}_\sigma^\kappa \mid \sigma \in \mathcal{L}_{(\kappa^{<\lambda})+\lambda}\}$.*
- (c) *Assume AC. Then $\mathbf{B}_{\text{cof}(\kappa)+1}^e(\mathcal{B}_b) \cap \text{Inv}_{\mathcal{L}}^\kappa \subseteq \{\text{Mod}_\sigma^\kappa \mid \sigma \in \mathcal{L}_{(\kappa^{<\kappa})+\kappa}\}$.*

PROPOSITION 8.14 (AC). *Suppose κ is regular.*

- (a) $\mathbf{B}_{\kappa+1}(\tau_p) \cap \text{Inv}_{\mathcal{L}}^\kappa \subseteq \{\text{Mod}_\sigma^\kappa \mid \sigma \in \mathcal{L}_{(\kappa^{<\kappa})+\kappa}\}$.
- (b) *Let $\omega < \lambda < \kappa$ be such that $\kappa^{<\lambda} = \kappa$. Then $\mathbf{B}_{\kappa+1}(\tau_\lambda) \cap \text{Inv}_{\mathcal{L}}^\kappa \subseteq \{\text{Mod}_\sigma^\kappa \mid \sigma \in \mathcal{L}_{(\kappa^{<\kappa})+\kappa}\}$.*
- (c) *Assume that $\kappa^{<\kappa} = \kappa$. Then $\mathbf{B}_{\kappa+1}(\tau_b) \cap \text{Inv}_{\mathcal{L}}^\kappa \subseteq \{\text{Mod}_\sigma^\kappa \mid \sigma \in \mathcal{L}_{\kappa+\kappa}\}$.*

The proofs of Propositions 8.13 and 8.14 are a careful refinement of an argument already implicit in the work of Vaught [Vau75] — see [Kec95, Proposition 16.9].

Recall from Section 6.1 that ${}^\kappa\kappa$ and its subspaces can be equipped with several topologies, namely $\tau_p = \tau_\omega, \tau_\mu$ with $\omega \leq \mu < \max(\text{cof}(\kappa)^+, \kappa)$, and τ_b . When κ is *regular* then $\tau_\kappa = \tau_b$ and $\widehat{\mathcal{B}}_\kappa = \widehat{\mathcal{B}}_b$ as agreed in (6.1), so that the notation τ_μ and $\widehat{\mathcal{B}}_\mu$ encompass all possibilities. If κ is *singular*, then this is not the case and the bounded topology and its canonical basis should be treated separately. This would cause a lot of notational inconveniences in the results that follow, so we stipulate the following:

CONVENTION 8.15. For the rest of this section, we agree that $\tau_\kappa = \tau_b$ and $\widehat{\mathcal{B}}_\kappa = \widehat{\mathcal{B}}_b$, independently of whether κ is regular or not. Thus the topologies and bases relevant for us are of the form τ_μ and $\widehat{\mathcal{B}}_\mu$ with μ in the set

$$(8.5) \quad \{\nu \in \text{Card} \mid \omega \leq \nu \leq \text{cof}(\kappa) \vee \nu = \kappa\}.$$

The group $\text{Sym}(\kappa)$ inherits the relative topology (denoted again by τ_μ), whose basic open sets can be written as

$$[s] := \widehat{\mathcal{N}}_{s^{-1}} \cap \text{Sym}(\kappa) = \{g \in \text{Sym}(\kappa) \mid s \subseteq g^{-1}\}$$

with $s \in {}^u(\kappa)$ (i.e. s is an injection from u to κ , see Section 2.1.2) and $u \in D(\mu)$, where

$$(8.6) \quad D(\mu) := \begin{cases} [\kappa]^{<\mu} & \text{if } \mu \neq \kappa, \\ \kappa & \text{if } \mu = \kappa. \end{cases}$$

Given a property φ for the elements of $\text{Sym}(\kappa)$ and a nonempty τ_μ -open set $U \subseteq \text{Sym}(\kappa)$, we write

$$\forall_\mu^* g \in U \varphi(g)$$

to abbreviate the statement:

$$\{g \in \text{Sym}(\kappa) \mid \varphi(g)\} \text{ is } \text{cof}(\mu)\text{-comeager in } U,$$

where the notion of μ -comeagerness is defined in Section 6.2. Next we define the (local) Vaught transform of a set $A \subseteq \text{Mod}_{\mathcal{L}}^\kappa$: given $u \in D(\mu)$, let

$$A_\mu^{*u} := \{(X, s) \in \text{Mod}_{\mathcal{L}}^\kappa \times {}^u(\kappa) \mid \forall_\mu^* g \in [s] (g.X \in A)\},$$

where $g.X$ is as in (7.5).

We are now going to prove Lemma 8.16, an analogue of [Kec95, Proposition 16.9], from which both Propositions 8.13 and 8.14 follow. The direct adaptation of the proof of [Kec95, Proposition 16.9] to our context yields a formula φ_u whose variables range in $\{\mathbf{v}_\alpha \mid \alpha < \kappa\}$, while in the logic $\mathcal{L}_{\nu\mu}$ of Lemma 8.16 we can use only variables from the (possibly smaller) set $\{\mathbf{v}_\alpha \mid \alpha < \mu\}$ (see Definition 8.1). Such an argument would yield a φ_u which is *essentially* what we require, but not quite an element of $\mathcal{L}_{\nu\mu}$. To overcome this purely technical difficulty, a somewhat artificial fragment $\mathcal{L}_{\nu\kappa}^\downarrow$ of $\mathcal{L}_{\nu\kappa}$ is introduced (see the beginning of the proof of Lemma 8.16). In order to define such fragment, we need a few preliminary definitions.

Let $D(\mu)$ be as in (8.6), and fix $u \in D(\mu)$. Set

$$u \downarrow \mu := \begin{cases} \text{the unique } \nu < \mu \text{ such that } u \in [\kappa]^\nu & \text{if } \mu \neq \kappa, \\ u & \text{if } \mu = \kappa \end{cases}$$

so that in either case $u \downarrow \mu \in [\mu]^{<\mu}$. Recalling from Section 2.1.2 that every element of $[\kappa]^{<\mu}$ is identified with its enumerating function, for any $s \in {}^u(\kappa)$ let

$$s \downarrow \mu := \begin{cases} s \circ u & \text{if } \mu \neq \kappa, \\ s & \text{if } \mu = \kappa, \end{cases}$$

so that in either case $s \downarrow \mu \in \bigcup_{\nu < \mu} {}^\nu(\kappa)$. Notice that when $u = s = \emptyset$ we have $u \downarrow \mu = s \downarrow \mu = \emptyset$ (for any cardinal μ as above).

LEMMA 8.16. *Suppose $\lambda \leq \mu$ both belong to the set in (8.5), and endow the spaces $\text{Sym}(\kappa)$ and $\text{Mod}_{\mathcal{L}}^\kappa$ with the topologies τ_μ and τ_λ , respectively. Let $A \subseteq \text{Mod}_{\mathcal{L}}^\kappa$ be in $\mathbf{B}_{\text{cof}(\mu)+1}^e(\mathcal{B}_\lambda)$, and let $u \in D(\mu)$. Let $\nu := |^{<\mu}\kappa|^+$, assuming AC if $\mu > \omega$. Then there is some $\mathcal{L}_{\nu\mu}$ -formula $\varphi_u(\mathbf{v}_{u \downarrow \mu})$ such that for every $X \in \text{Mod}_{\mathcal{L}}^\kappa$ and $s \in {}^u(\kappa)$*

$$(X, s) \in A_\mu^{*u} \Leftrightarrow (X, s \downarrow \mu) \in M_{\varphi_u, u \downarrow \mu},$$

where $M_{\varphi,u}$ is as in (8.2).

PROOF. Let $\mathcal{L}_{\nu\kappa}^{\downarrow\mu}$ be the fragment of $\mathcal{L}_{\nu\kappa}$ obtained by adding to Definition 8.1 the following restrictions:

- the generalized conjunction of the formulæ φ_α (for $\alpha < \nu < \nu$) may be formed only when the total number of variables occurring free in some of the φ_α 's is $< \mu$;
- a generalized existential quantification $\exists v_u \varphi$ may be formed only when $u \in [\kappa]^{<\mu}$.

Equivalently, $\mathcal{L}_{\nu\kappa}^{\downarrow\mu}$ can be defined similarly to the logic $\mathcal{L}_{\nu\mu}$ except that we may use variables from a longer list $\langle v_\alpha \mid \alpha < \kappa \rangle$ of length κ instead of using μ -many variables. (When $\mu = \kappa$ we get $\mathcal{L}_{\nu\kappa}^{\downarrow\mu} = \mathcal{L}_{\nu\kappa}$.) It follows in particular that each formula in $\mathcal{L}_{\nu\kappa}^{\downarrow\mu}$ has $< \mu$ -many free variables occurring in it. In order to simplify the notation, for the rest of the proof we write $\mathcal{L}_{\nu\kappa}^{\downarrow}$, $u\downarrow$, and $s\downarrow$ instead of $\mathcal{L}_{\nu\kappa}^{\downarrow\mu}$, $u\downarrow\mu$ and $s\downarrow\mu$.

By a suitable variable substitution, any $\psi'(v_u) \in \mathcal{L}_{\nu\kappa}^{\downarrow}$ can be easily transformed into a corresponding $\psi(v_{u\downarrow}) \in \mathcal{L}_{\nu\mu}$ such that for all $(X, s) \in \text{Mod}_{\mathcal{L}}^\kappa \times {}^u(\kappa)$

$$(X, s) \in M_{\psi',u} \Leftrightarrow (X, s\downarrow) \in M_{\psi,u\downarrow}.$$

Thus it is enough to find, for any given A as in its hypotheses of the lemma, an $\mathcal{L}_{\nu\kappa}^{\downarrow}$ -formula $\varphi'_u(v_u)$ such that $A_\mu^{*u} = M_{\varphi'_u,u}$: setting $\varphi'_u := G(\emptyset, u)$ in the following construction we will be done.

Let (T, ϕ) be a $\text{cof}(\mu) + 1$ -Borel code for $A \subseteq \text{Mod}_{\mathcal{L}}^\kappa$ such that $\phi(t) \in \mathcal{B}_\lambda(\text{Mod}_{\mathcal{L}}^\kappa)$ for every terminal node $t \in T$. We shall define a map

$$G: T \times D(\mu) \rightarrow \mathcal{L}_{\nu\kappa}^{\downarrow}$$

such that $\text{Fv}(G(t, u)) = u$ and

$$\phi(t)_\mu^{*u} = M_{G(t,u),u}.$$

The map G is defined inductively on the well-founded relation

$$(t, u) \leq (t', u') \Leftrightarrow t' \text{ precedes } t \text{ in the ordering of } T,$$

using that $(\text{Sym}(\kappa), \tau_\mu)$ is $\text{cof}(\mu)$ -Baire (Theorem 6.12).

Case 1: t is a terminal node of T , so that $\phi(t) \in \mathcal{B}_\lambda(\text{Mod}_{\mathcal{L}}^\kappa)$. Then it is easy to check that there are²⁷ $\theta(v_w) \in \mathcal{L}_{\nu\kappa}^{\downarrow}$ with $w \in D(\mu)$, and $h \in {}^w(\kappa)$, such that

$$\phi(t) = \{X \in \text{Mod}_{\mathcal{L}}^\kappa \mid X \models \theta[h]\}.$$

Then for every $u \in D(\mu)$ we have

$$(8.7) \quad \begin{aligned} (X, s) \in \phi(t)_\mu^{*u} &\Leftrightarrow s \in {}^u(\kappa) \wedge \forall_\mu^* g \in [s] (g.X \models \theta[h]) \\ &\Leftrightarrow s \in {}^u(\kappa) \wedge \forall_\mu^* g \in [s] (X \models \theta[g^{-1} \circ h]). \end{aligned}$$

Let $w' := \{h(\alpha) \mid \alpha \in w\}$, and let $\theta'(v_{w'})$ be the formula obtained from $\theta(v_w)$ by substituting each (free) occurrence of v_α in $\theta(v_w)$ with $v_{h(\alpha)}$ (for each $\alpha \in w$). Now there are two cases:

- if $w' \subseteq u$ then since $g \in [s] \Leftrightarrow s \subseteq g^{-1}$ we have $g^{-1} \circ h = s \circ h = s \upharpoonright w'$, so $\phi(t)_\mu^{*u} = M_{\varphi,u}$ with $\varphi(v_u) \in \mathcal{L}_{\nu\kappa}^{\downarrow}$ being the formula

$$\bigwedge_{\substack{\alpha, \beta \in u \\ \alpha < \beta}} v_\alpha \neq v_\beta \wedge \theta'(v_{w'}).$$

In fact if $(X, s) \in \phi(t)_\mu^{*u}$ then by (8.7) there is $g \in [s]$ such that $X \models \theta[g^{-1} \circ h]$, and hence $X \models \theta[s \circ h]$. From this it follows that $X \models \varphi[s]$, i.e. $(X, s) \in M_{\varphi,u}$. Conversely, if $(X, s) \in M_{\varphi,u}$ then $X \models \varphi[s]$. Therefore $X \models \theta[g^{-1} \circ h]$ for every $g \in [s]$, and hence $(X, s) \in \phi(t)_\mu^{*u}$ since $[s]$ is trivially $\text{cof}(\mu)$ -comeager in itself.

²⁷In fact θ is a Boolean combination of atomic formulæ of \mathcal{L} using conjunctions and disjunctions of size $< \lambda$.

- if $w' \not\subseteq u$, let $w'' \in D(\mu)$ be smallest such that $u \cup w' \subseteq w''$ (i.e. $w'' := u \cup w'$ if $\lambda = \mu = \kappa$ or $\lambda \leq \mu < \kappa$, while $w'' := \sup \{\alpha < \kappa \mid \alpha \in u \cup w'\}$ if $\lambda < \mu = \kappa$). As every $\text{cof}(\mu)$ -comeager subset of $[s]$ must intersect every $[r]$ with $r \in {}^{w''}(\kappa)$ and $s \subseteq r$, and since $(\text{Sym}(\kappa), \tau_\mu)$ is $\text{cof}(\mu)$ -Baire, then

$$\forall_\mu^* g \in [s] \left(X \models \theta[g^{-1} \circ h] \right) \Leftrightarrow \forall r \in {}^{w''}(\kappa) (r \supseteq s \Rightarrow X \models \theta'[r \restriction w']).$$

Arguing as in the previous case, $\phi(t)_\mu^{*u} = M_{\varphi, u}$ with $\varphi(v_u) \in \mathcal{L}_{\nu_\kappa}^\perp$ being the formula

$$\left(\bigwedge_{\substack{\alpha, \beta \in u \\ \alpha < \beta}} v_\alpha \neq v_\beta \right) \wedge \forall \langle v_\alpha \mid \alpha \in w'' \setminus u \rangle \left(\bigwedge_{\substack{\alpha, \beta \in w'' \\ \alpha < \beta}} v_\alpha \neq v_\beta \Rightarrow \theta'(v_{w'}) \right).$$

In either case let $G(t, u) := \varphi(v_u)$.

Case 2: otherwise. Let

$$\text{Succ}(t) := \{t' \in T \mid t' \text{ is an immediate successor of } t\},$$

which is a set of size $\leq \text{cof}(\mu)$. By inductive hypothesis, for every $t' \in \text{Succ}(t)$ and every $w \in D(\mu)$ we have that $G(t', w) \in \mathcal{L}_{\nu_\kappa}^\perp$ and $\phi(t')_\mu^{*w} = M_{G(t', w), w}$. As $(\text{Sym}(\kappa), \tau_\mu)$ is $\text{cof}(\mu)$ -Baire, then

$$\left(\bigcap_{t' \in \text{Succ}(t)} \phi(t') \right)_\mu^{*w} = \bigcap_{t' \in \text{Succ}(t)} \phi(t')_\mu^{*w} = \bigcap_{t' \in \text{Succ}(t)} M_{G(t', w), w} = M_{\psi_w, w},$$

where $\psi_w(v_w)$ is $\bigwedge_{t' \in \text{Succ}(t)} G(t', w)$. Given $X \in \text{Mod}_{\mathcal{L}}^\kappa$, the map

$$\text{Sym}(\kappa) \rightarrow \text{Mod}_{\mathcal{L}}^\kappa, \quad g \mapsto g.X$$

is continuous when both spaces are endowed with the topology τ_λ , and hence it is also continuous as a function between $(\text{Sym}(\kappa), \tau_\mu)$ and $(\text{Mod}_{\mathcal{L}}^\kappa, \tau_\lambda)$, as τ_μ refines τ_λ . Since $\phi(t) \in \mathbf{B}_{\text{cof}(\mu)+1}^e(\tau_\lambda) \subseteq \mathbf{B}_{\text{cof}(\mu)+1}(\tau_\lambda)$ and $\mathbf{B}_{\text{cof}(\mu)+1}$ is closed under continuous preimages, we get that

$$(8.8) \quad \{g \in \text{Sym}(\kappa) \mid g.X \in \phi(t)\} \in \mathbf{B}_{\text{cof}(\mu)+1}(\text{Sym}(\kappa), \tau_\mu).$$

Now fix an arbitrary $u \in D(\mu)$ in order to define $G(t, u)$. By (8.8) and Proposition 6.14, for every $s \in {}^u(\kappa)$ the set $\{g \in [s] \mid g.X \in \phi(t)\}$ has the $\text{cof}(\mu)$ -Baire property. Using Proposition 6.15,

$$\begin{aligned} (X, s) \in (\phi(t))_\mu^{*u} &\Leftrightarrow (X, s) \in (\text{Mod}_{\mathcal{L}}^\kappa \setminus \bigcap_{t' \in \text{Succ}(t)} \phi(t'))_\mu^{*u} \\ &\Leftrightarrow \forall u \subseteq w \in D(\mu) \forall s \subseteq r \in {}^w(\kappa) \left[(X, r) \notin \left(\bigcap_{t' \in \text{Succ}(t)} \phi(t') \right)_\mu^{*w} \right] \\ &\Leftrightarrow \forall u \subseteq w \in D(\mu) \forall s \subseteq r \in {}^w(\kappa) \left[(X, r) \notin M_{\psi_w, w} \right], \end{aligned}$$

so that $\phi(t)_\mu^{*u} = M_{\varphi, u}$ where $\varphi(v_u)$ is the formula

$$\left(\bigwedge_{\substack{\alpha, \beta \in u \\ \alpha < \beta}} (v_\alpha \neq v_\beta) \right) \wedge \bigwedge_{u \subseteq w \in D(\mu)} \forall \langle v_\alpha \mid \alpha \in w \setminus u \rangle \left[\left(\bigwedge_{\substack{\alpha, \beta \in w \\ \alpha < \beta}} v_\alpha \neq v_\beta \right) \Rightarrow \neg \psi_w(v_w) \right].$$

Therefore it is enough to put $G(t, u) := \varphi(v_u)$. \square

Varying the parameters μ and λ in Lemma 8.16 and taking $u = \emptyset$, we can now prove Propositions 8.13 and 8.14.

PROOF OF PROPOSITION 8.13. First notice that if $A \subseteq \text{Mod}_{\mathcal{L}}^\kappa$ is invariant under isomorphism, then for every infinite cardinal μ belonging to the set in (8.5) and every $X \in \text{Mod}_{\mathcal{L}}^\kappa$ we have $X \in A \Leftrightarrow (X, \emptyset) \in A_\mu^{*\emptyset}$ by Theorem 6.12. Recall also from (8.4) that for every sentence $\sigma \in \mathcal{L}_{|<\mu_\kappa|+\mu}$ and every $X \in \text{Mod}_{\mathcal{L}}^\kappa$ we have $(X, \emptyset) \in M_{\sigma, \emptyset} \Leftrightarrow X \in \text{Mod}_\sigma^\kappa$, and that $\emptyset^{\downarrow \mu} = \emptyset$. Then it is enough to set $\mu = \lambda$ and $u = \emptyset$ in Lemma 8.16, and then further set $\lambda = \omega$ to get (a) and $\lambda = \kappa$ to get (c). \square

PROOF OF PROPOSITION 8.14. The proof is analogous to that of Proposition 8.13: it is enough to set $\mu = \kappa$ and $u = \emptyset$ in Lemma 8.16, and then further set $\lambda = \omega$ for (a) and $\lambda = \kappa$ for (c). The assumption $\kappa^{<\lambda} = \kappa$ in part (b) guarantees that $|\mathcal{B}_\lambda| = \kappa$, so that $\tau_\lambda \subseteq \mathbf{B}_{\kappa+1}^e(\mathcal{B}_\lambda)$, and therefore $\mathbf{B}_{\kappa+1}(\tau_\lambda) = \mathbf{B}_{\kappa+1}^e(\mathcal{B}_\lambda)$. Similarly $\kappa^{<\kappa} = \kappa$ in part (c) ensures that $\mathbf{B}_{\kappa+1}(\tau_b) = \mathbf{B}_{\kappa+1}^e(\mathcal{B}_b)$. \square

REMARK 8.17. As for Corollary 8.11 (see Remark 8.12), if we drop the assumption $\kappa^{<\kappa} = \kappa$ then also Proposition 8.14(c) may fail. Work in ZFC. As observed in [MR13, Theorem 4.4], if μ is regular and $\mu < \kappa$, then there are $2^{(2^\mu)}$ -many τ_b -open subsets of $\text{Mod}_{\mathcal{L}}^\kappa$ invariant under isomorphism, while there are $(\kappa^{<\kappa})^\kappa = 2^\kappa$ formulæ in $\mathcal{L}_{\kappa+\kappa}$. Thus if there is e.g. a regular $\mu < \kappa$ such that $2^{(2^\mu)} > 2^\kappa$ (which can happen if $\kappa^{<\kappa} > \kappa$), then there is also an (effective) invariant $\kappa + 1$ -Borel subset of $\text{Mod}_{\mathcal{L}}^\kappa$ which cannot be of the form Mod_σ^κ for σ an $\mathcal{L}_{\kappa+\kappa}$ -sentence. In particular, if $2^{\aleph_0} = 2^{\aleph_1}$, as it is the case in models of forcing axioms like MA_{ω_1} or PFA, then there is an invariant τ_b -open subset of $\text{Mod}_{\mathcal{L}}^{\omega_1}$ which is not of the form $\text{Mod}_\sigma^{\omega_1}$ for any $\sigma \in \mathcal{L}_{\omega_2\omega_1}$ ([MR13, Corollary 4.6]). A similar argument shows that also Proposition 8.14(b) may fail if we do not assume that $\kappa^{<\lambda} = \kappa$.

9. κ -Souslin sets

9.1. Basic facts. The following definition generalizes (in a different direction from the one considered in Section 7.1) the notion of an analytic subset of a Polish space.

DEFINITION 9.1. Let κ be an infinite cardinal, ${}^\omega\kappa$ be endowed with the product of the discrete topology on κ , and X be a Polish space. A set $A \subseteq X$ is called κ -**Souslin** if it is a continuous image of a closed subset of ${}^\omega\kappa$, and is called ∞ -**Souslin** if there is an infinite cardinal κ such that A is κ -Souslin. The class of all κ -Souslin sets is denoted by $\mathcal{S}(\kappa)$, and $\mathcal{S}(\infty) := \bigcup_{\kappa \in \text{Card}} \mathcal{S}(\kappa)$ is the collection of all ∞ -Souslin sets.

Thus $\mathcal{S}(\omega) = \Sigma_1^1$ and $\Sigma_2^1 \subseteq \mathcal{S}(\omega_1)$, but the reverse inclusion $\mathcal{S}(\omega_1) \subseteq \Sigma_2^1$ depends on the axioms we assume: it is true in models of $\text{AD} + \text{DC}$, but it is false in models of $\text{AC} + \text{CH}$. Under choice every subset of a Polish space is 2^{\aleph_0} -Souslin (see Proposition 9.14), so the notion of an ∞ -Souslin set makes sense only if we work in models where AC fails.

The class $\mathcal{S}(\kappa)$ is a hereditary boldface pointclass, it is closed under countable unions and countable intersections (assuming AC_ω), it contains all Borel sets, and it is closed under images and preimages of Borel functions (Lemma 9.7). In particular, $\mathcal{S}(\kappa) \supseteq \Sigma_1^1$ for every infinite κ . Moreover, every κ -Souslin set is automatically κ' -Souslin for every $\kappa' \geq \kappa$, so that $\mathcal{S}(\kappa) \subseteq \mathcal{S}(\kappa')$ (although it is not true in general that $\kappa < \kappa' \Rightarrow \mathcal{S}(\kappa) \subset \mathcal{S}(\kappa')$: by Corollary 9.15 this is true under AC for all $\kappa' \leq 2^{\aleph_0}$, but under $\text{AD} + \text{DC}$ we e.g. have $\mathcal{S}(\omega_1) = \mathcal{S}(\omega_2)$ — see Section 9.4). However, it is maybe worth noticing that the pointclasses $\mathcal{S}(\kappa)$, $\kappa \in \text{Card}$, may not form a well behaved hierarchy in models of AC as it may happen that $\check{\mathcal{S}}(\kappa) \not\subseteq \mathcal{S}(\kappa')$ for $\omega_1 \leq \kappa < \kappa'$ (see Remark 9.17). This pathology is absent in models of AD , where we have

$$\mathcal{S}(\kappa) \cup \check{\mathcal{S}}(\kappa) \subseteq \mathcal{S}(\kappa') \cap \check{\mathcal{S}}(\kappa')$$

for all Souslin cardinals $\kappa < \kappa'$ (see Definition 9.11).

REMARK 9.2. Using standard arguments and the closure properties of $\mathcal{S}(\kappa)$ mentioned above, the notion of κ -Souslin set may be reformulated in several ways. For any subset A of a Polish spaces X and any infinite cardinal κ the following are equivalent:

- A is κ -Souslin;
- A is either empty or a continuous image of ${}^\omega\kappa$;
- A is a continuous image of a Borel subset of ${}^\omega\kappa$;
- $A = pF$ for some closed $F \subseteq X \times {}^\omega\kappa$;

- $A = pB$ for some Borel $B \subseteq X \times {}^\omega\kappa$,

where p is the projection map defined in (2.8) on page 18.

Since $\mathbf{S}(\kappa)$ is closed under countable unions, κ -Souslinness becomes trivial when considering a countable Polish space X , as any $A \subseteq X$ is countable, and therefore $A \in \mathbf{S}(\omega) \subseteq \mathbf{S}(\kappa)$ for every infinite cardinal κ . Moreover, any two uncountable Polish spaces are Borel isomorphic, the closure under images and preimages of Borel functions of $\mathbf{S}(\kappa)$ yields that it is enough to study κ -Souslin subsets of some specific uncountable Polish space. *Therefore, unless otherwise specified, from now on we confine our analysis to κ -Souslin subsets of (countable products of) ${}^\omega 2$.* The choice of these canonical spaces is motivated by the fact that the κ -Souslin subsets of $({}^\omega 2)^N$ (for $1 \leq N \leq \omega$) admit a particularly nice representation in terms of projections of the trees introduced in the following

NOTATION. Recall from (2.7) on page 18 that $\text{Tr}(Y)$ is the set of all descriptive set-theoretic trees (of height $\leq \omega$) on Y . When $Y = \underbrace{(2 \times 2 \times \cdots)}_N \times \kappa$ we often write $\text{Tr}(N; 2, \kappa)$ instead of $\text{Tr}(Y)$.

REMARK 9.3. As $Y := \underbrace{(2 \times 2 \times \cdots)}_N \times \kappa$ has size κ , the elements of $\text{Tr}(Y)$ can be identified (in ZF) with elements of ${}^\kappa 2$.

The usual representation of analytic (i.e. ω -Souslin) sets (see [Kec95, Proposition 25.2]) extends to an arbitrary cardinal κ , so that for every $A \subseteq ({}^\omega 2)^N$

$$A \in \mathbf{S}(\kappa) \Leftrightarrow \exists T \in \text{Tr}(N; 2, \kappa) (A = p[T]).$$

A similar tree representation holds for κ -Souslin subsets of ${}^\omega\omega$, or more generally, of any countable power ${}^\omega X$ of a discrete countable set X . As hinted in Section 2.5.3, using such representation and the notion of leftmost branch through trees on well-orderable sets, one can easily reformulate the notion of a κ -Souslin set in terms of scales (see e.g. [Jac10, Lemma 2.5]):

FACT 9.4. *A set $A \subseteq {}^\omega X$ is κ -Souslin if and only if it admits a scale all of whose norms map into κ .*

The fact that a set $A \subseteq ({}^\omega 2)^N$ is κ -Souslin gives us some structural information on it: for example, we have the following property, which is meaningful whenever κ is small enough compared to ${}^\omega 2$ (see [Mos09, Theorem 2C.2]).

PROPOSITION 9.5. *Let κ be an infinite cardinal and let $A \subseteq ({}^\omega 2)^N$. If $A \in \mathbf{S}(\kappa)$ then A has the κ -**Perfect Set Property** (κ -PSP for short): either A has at most κ -many elements, or else it contains a perfect set (equivalently, a homeomorphic copy of ${}^\omega 2$).*

This property is used in Section 9.3 to show that the pointclass $\mathbf{S}(\kappa)$ is not trivial for small κ . Another way to obtain nontrivial pointclasses is to relativize κ -Souslinness to some inner model, as explained in the following remark — this approach is exploited in Section 14.3.

REMARKS 9.6. (i) If W is an inner model and κ is a cardinal in it, by absoluteness of existence of infinite branches, the relativization²⁸ of $\mathbf{S}(\kappa)$ to W is

$$(\mathbf{S}(\kappa))^W = \{A \cap ({}^\omega 2)^W \mid A \in \mathbf{S}_W(\kappa)\} \supseteq \mathbf{S}_W(\kappa) \cap \mathcal{P}(({}^\omega 2)^W)$$

where²⁹

$$\mathbf{S}_W(\kappa) := \{p[T] \mid T \text{ is a tree on } 2 \times \kappa \text{ belonging to } W\}.$$

²⁸To simplify the notation, we consider here only subsets of ${}^\omega 2$: the generalization of this notion to subsets of countable products $({}^\omega 2)^N$ (and to arbitrary Polish spaces) is straightforward.

²⁹In the definition of $\mathbf{S}_W(\kappa)$, the tree T must belong to W but its projection $p[T]$ is computed in V .

In particular, if ${}^\omega 2 \subseteq W$ (and hence $L(\mathbb{R}) \subseteq W$) then

$$(\mathcal{S}(\kappa))^W = \mathcal{S}_W(\kappa).$$

- (ii) If ${}^\omega 2 \subseteq W$ (equivalently, $L(\mathbb{R}) \subseteq W$), then $\mathcal{S}_W(\kappa)$ is a boldface pointclass in V because $L(\mathbb{R})$ contains all projective sets and all continuous functions. Moreover, $\mathcal{S}_W(\kappa)$ contains all Borel sets and is closed under continuous images, so that $\Sigma_1^1 \subseteq \mathcal{S}_W(\kappa) \subseteq \mathcal{S}(\kappa)$.

9.2. More on Souslin sets and Souslin cardinals. The following lemma collects some well-known facts on $\mathcal{S}(\kappa)$.

LEMMA 9.7 (\mathbf{AC}_ω). *Let κ be an infinite cardinal.*

- (a) $\mathcal{S}(\kappa)$ is a hereditary boldface pointclass containing all closed and open sets, closed under countable unions and countable intersections (and hence containing all Borel sets), and closed under projections (equivalently, under continuous images). In particular, $\mathcal{S}(\kappa)$ contains all analytic sets.
- (b) $\mathcal{S}(\kappa)$ is closed under images and preimages of (partial) functions with κ -Souslin graph.

Notice that the assumption \mathbf{AC}_ω can be relaxed to $\mathbf{AC}_\omega(\mathbb{R})$ if \mathbf{AD} is assumed and $\kappa < \Theta$. Moreover, since a function between two Polish spaces is Borel if and only if its graph is ω -Souslin (i.e. analytic), Lemma 9.7 implies that every pointclass $\mathcal{S}(\kappa)$ is closed under Borel images and Borel preimages.

PROOF. The proof of (a) is standard — see [Mos09, Theorem 2B.2].

(b) Let X, Y be Polish spaces, $A \subseteq X$ be κ -Souslin, and f be a partial function from X to Y with κ -Souslin graph. We must show that

$$B := \{y \in Y \mid \exists x \in A \cap \text{dom}(f) (f(x) = y)\} \in \mathcal{S}(\kappa).$$

We can assume without loss of generality that both A and $\text{graph}(f)$ are nonempty, so let $g: {}^\omega \kappa \rightarrow X$ and $h: {}^\omega \kappa \rightarrow X \times Y$ be continuous surjections onto A and $\text{graph}(f)$, respectively. For $i = 0, 1$, let $\pi_i: {}^\omega \kappa \rightarrow {}^\omega \kappa$, $s \mapsto \langle s(2n+i) \mid n \in \omega \rangle$, and let π_X and π_Y be the projections of $X \times Y$ onto the spaces X and Y , respectively. Set

$$C := \{s \in {}^\omega \kappa \mid g(\pi_0(s)) = \pi_X(h(\pi_1(s)))\}.$$

Since all the functions involved in its definition are continuous and X is Hausdorff, $C \subseteq {}^\omega \kappa$ is a closed set, and hence there is a continuous $r: {}^\omega \kappa \rightarrow C$ such that $r \upharpoonright C$ is the identity map by [Kec95, Proposition 2.8]. Then $\pi_Y \circ h \circ \pi_1 \circ r: {}^\omega \kappa \rightarrow B$ is continuous, whence $B \in \mathcal{S}(\kappa)$. \square

REMARK 9.8. If κ is an infinite cardinal and we assume \mathbf{AC}_κ , then $\mathcal{S}(\kappa)$ is further closed under unions of length (at most) κ . This is because if $\langle A_\alpha \mid \alpha < \kappa \rangle$ is a sequence of κ -Souslin subsets of a Polish space X , using \mathbf{AC}_κ we can choose for each $\alpha < \kappa$ a continuous surjection $p_\alpha: {}^\omega \kappa \rightarrow A_\alpha \subseteq X$. The surjection

$${}^\omega \kappa \rightarrow \bigcup_{\alpha < \kappa} A_\alpha, \quad \langle \alpha_0, \alpha_1, \alpha_2, \dots \rangle \mapsto p_{\alpha_0}(\langle \alpha_1, \alpha_2, \dots \rangle)$$

is continuous, whence $\bigcup_{\alpha < \kappa} A_\alpha \in \mathcal{S}(\kappa)$. In fact, assuming \mathbf{AC} one can further show that for every $0 \neq n \in \omega$ and every subset A of a Polish space X , A is ω_n -Souslin if and only if A is a union of \aleph_n -many Borel sets (see e.g. [Kan03, Proposition 13.13(f)]).

The following lemma collects some classical facts relating κ -Souslinity to $\kappa + 1$ -Borelness.

LEMMA 9.9 (Folklore). *Every $A \in \mathcal{S}(\kappa)$ is effective $\kappa^+ + 1$ -Borel (in fact, A is an effective κ^+ -union of effective $\kappa + 1$ -Borel sets), and if κ has uncountable cofinality then A is an effective κ -union of effective κ -Borel sets (thus it is effective $\kappa + 1$ -Borel). In particular, $\mathcal{S}(\kappa) \subseteq \mathbf{B}_{\kappa^++1}^{(e)}$ and*

if κ has uncountable cofinality then also $\mathcal{S}(\kappa) \subseteq \mathbf{B}_{\kappa+1}^{(e)}$. Moreover, $\Delta_{\mathcal{S}(\kappa)} \subseteq \mathbf{B}_{\kappa+1}$ (independently of κ).

PROOF. Use the characterization of κ -Souslin subsets of ${}^\omega 2$ in terms of projection of trees in $\text{Tr}(2 \times \kappa)$ (see [Jac08, Lemma 2.12]). For the part concerning $\Delta_{\mathcal{S}(\kappa)}$, use the classical Lusin separation argument (see [Mos09, Theorem 2E.2]). \square

PROPOSITION 9.10. *Suppose that there is an $\mathcal{S}(\kappa)$ -code for κ , so that it makes sense to speak of a $\mathcal{S}(\kappa)$ -in-the-codes Borel function from ${}^\omega 2$ to ${}^\kappa 2$. Then every $\mathcal{S}(\kappa)$ -in-the-codes function $f: {}^\omega 2 \rightarrow {}^\kappa 2$ is weakly $\kappa + 1$ -Borel.*

PROOF. By Lemma 9.7, $\mathcal{S}(\kappa)$ is closed under projections and finite intersections (notice that for these closure properties AC_ω is not needed). By (5.1) on page 36, for every $\alpha < \kappa$ and every $i \in \{0, 1\}$, the set $f^{-1}(\tilde{N}_{\alpha,i}^\kappa)$ is in $\Delta_{\mathcal{S}(\kappa)}$. Since by Lemma 9.9 every set in $\Delta_{\mathcal{S}(\kappa)}$ is $\kappa + 1$ -Borel, this means that each $f^{-1}(\tilde{N}_{\alpha,i}^\kappa)$ is $\kappa + 1$ -Borel. Since the sets $\tilde{N}_{\alpha,i}^\kappa$ generates (by taking finite intersections) the canonical basis \mathcal{B}_p for the product topology τ_p on ${}^\kappa 2$, and since \mathcal{B}_p , being in bijection with $\mathbf{Fn}(\kappa, 2; \omega)$, is a well-orderable set size κ , this implies that $f^{-1}(U)$ is $\kappa + 1$ -Borel for every τ_p -open $U \subseteq {}^\kappa 2$, i.e. that f is weakly $\kappa + 1$ -Borel. \square

DEFINITION 9.11. A cardinal κ is a **Souslin cardinal** if $\mathcal{S}(\kappa) \setminus \bigcup_{\lambda < \kappa} \mathcal{S}(\lambda) \neq \emptyset$, i.e. if there is a κ -Souslin set which is not λ -Souslin for any $\lambda < \kappa$. Let

$$\Xi := \sup \{ \kappa \in \text{Card} \mid \kappa \text{ is a Souslin cardinal} \}.$$

By definition $\mathcal{S}(\infty) = \mathcal{S}(\Xi)$, and by Lemma 9.9 $\mathcal{S}(\infty) \subseteq \mathbf{B}_\infty$. The converse inclusion may fail, for example $\text{AD} + \text{V} = \text{L}(\mathbb{R})$ implies that $\mathcal{S}(\infty) = \Sigma_1^2$ and that every set of reals is ∞ -Borel. The next folklore result shows that $\Xi \leq \Theta$.

LEMMA 9.12. *If κ is a Souslin cardinal, then $\kappa < \Theta$. In particular, Θ is not a Souslin cardinal.*

PROOF. Let $A \subseteq {}^\omega \omega$ witness that κ is a Souslin cardinal. Then by Fact 9.4 there is a scale $\langle \rho_n \mid n \in \omega \rangle$ on A such that each ρ_n maps into a cardinal $\kappa_n \leq \kappa$. Let $\lambda := \sup_{n \in \omega} \kappa_n \leq \kappa$. If $\lambda < \kappa$ then A would be λ -Souslin by Fact 9.4 again, contradicting the choice of A : then $\lambda = \kappa$, and the map $\varphi: {}^\omega \omega \rightarrow \kappa$ sending each $n^\frown x$ to $\rho_n(x)$ if $x \in A$ and to 0 otherwise is surjective, whence $\kappa < \Theta$. \square

As $\Sigma_2^1 \setminus \Sigma_1^1 \neq \emptyset$ and as $\Sigma_1^1 = \mathcal{S}(\omega)$ and $\Sigma_2^1 \subseteq \mathcal{S}(\omega_1)$, then ω_1 is a Souslin cardinal. The existence of a Souslin cardinal greater than ω_1 , and whether Ξ is a Souslin cardinal (and hence the largest Souslin cardinal) or, equivalently, whether $\mathcal{S}(\Xi) \neq \bigcup_{\kappa < \Xi} \mathcal{S}(\kappa)$, both depend on the underlying set-theoretic assumptions — see Sections 9.3 and 9.4.

9.3. Souslin sets and cardinals in models with choice. Under choice $\Xi \leq 2^{\aleph_0}$ because every set of reals is trivially 2^{\aleph_0} -Souslin (Proposition 9.14). Therefore CH implies that $\Xi = \omega_1$ (and hence Ξ is a Souslin cardinal) and that $\mathcal{S}(\Xi) = \mathcal{S}(\omega_1) = \mathcal{P}({}^\omega 2)$, so the pointclass $\mathcal{S}(\omega_1)$ and the notion of ω_1 -Souslin set are uninteresting in this context. Assuming instead $\neg\text{CH}$, the condition $\kappa < 2^{\aleph_0}$ guarantees that $\mathcal{S}(\kappa)$ is a proper pointclass.

PROPOSITION 9.13 (AC). $\mathcal{P}({}^\omega 2) \setminus \bigcup_{\kappa < 2^{\aleph_0}} \mathcal{S}(\kappa) \neq \emptyset$, so 2^{\aleph_0} is a Souslin cardinal.

PROOF. The standard construction of a Bernstein set [Kec95, p. 38] yields a set $B \subseteq {}^\omega 2$ of size 2^{\aleph_0} which does not contain any perfect set, and hence $B \notin \bigcup_{\kappa < 2^{\aleph_0}} \mathcal{S}(\kappa)$ by Proposition 9.5. \square

The next result shows that in models of AC all cardinals smaller than the continuum are Souslin cardinals, whence

$$\forall \omega \leq \kappa, \kappa' < 2^{\aleph_0} \quad (\kappa \leq \kappa' \Leftrightarrow \mathcal{S}(\kappa) \subseteq \mathcal{S}(\kappa')).$$

PROPOSITION 9.14. *Let κ be an infinite cardinal. If $A \subseteq {}^\omega 2$ is an infinite set of size κ , then $A \in \mathcal{S}(\kappa)$. If moreover $|{}^\omega 2| \not\leq \kappa$, then $A \in \mathcal{S}(\kappa) \setminus \bigcup_{\lambda < \kappa} \mathcal{S}(\lambda)$. In particular, if κ is a small cardinal such that $|{}^\omega 2| \not\leq \kappa$ and we assume $\text{AC}_\kappa(\mathbb{R})$, then κ is a Souslin cardinal.*

PROOF. Given $A = \{x_\alpha \mid \alpha < \kappa\} \subseteq {}^\omega 2$, let

$$T := \{(x_\alpha \upharpoonright n, \alpha^{(n)}) \mid \alpha < \kappa, n \in \omega\},$$

where $\alpha^{(n)}$ is the sequence formed by n -many α 's. It is straightforward to check that $A = p[T]$, so that $A \in \mathcal{S}(\kappa)$.

Assume now that $|{}^\omega 2| \not\leq \kappa$, so that A cannot contain a perfect set. If $A \in \mathcal{S}(\lambda)$ for some infinite $\lambda < \kappa$, then the set A would have cardinality $\leq \lambda$ by the λ -PSP (Proposition 9.5), a contradiction. Thus $A \in \mathcal{S}(\kappa) \setminus \bigcup_{\lambda < \kappa} \mathcal{S}(\lambda)$.

Finally let κ be an infinite small cardinal, and let $p: {}^\omega 2 \rightarrow \kappa$ be a surjection. Then by $\text{AC}_\kappa(\mathbb{R})$ there is a choice function for the sequence $\langle p^{-1}(\alpha) \mid \alpha \in \kappa \rangle$, and the range A of such a function is a subset of ${}^\omega 2$ of cardinality κ : the result then follows from the first part of the proposition. \square

COROLLARY 9.15 (AC). *All infinite $\kappa \leq 2^{\aleph_0}$ are Souslin cardinals. In particular, $\Xi = 2^{\aleph_0}$.*

Thus under e.g. $\text{ZFC} + \text{PFA}$, the Souslin cardinals are exactly ω , ω_1 , and $\omega_2 = \Xi = 2^{\aleph_0}$. As for the cardinality of $\mathcal{S}(\kappa)$, notice that under choice $|\mathcal{S}(\kappa)| = 2^\kappa$. In fact $|\mathcal{S}(\kappa)| \leq 2^\kappa$ by Remark 9.3; for the other inequality use the fact that all subsets of a given $A \subseteq {}^\omega 2$ of size κ are κ -Souslin by Proposition 9.14. Thus it may happen that $|\mathcal{S}(\kappa)| < |\mathcal{S}(\kappa')|$ for some $\kappa < \kappa' \leq \Xi$.

The following proposition computes the value of $\delta_{\mathcal{S}(\kappa)}$ (for various $\kappa \leq 2^{\aleph_0}$) in models of AC, and should be contrasted with Proposition 9.25 providing analogous computations in the AD-context.

PROPOSITION 9.16 (AC). *Let $\kappa \leq 2^{\aleph_0}$ be an infinite cardinal.*

- (a) $\delta_{\mathcal{S}(\kappa)} \leq \kappa^+$.
- (b) $\delta_{\mathcal{S}(\omega)} = \omega_1$, $\delta_{\mathcal{S}(\omega_1)} = \omega_2$ and $\delta_{\mathcal{S}(2^{\aleph_0})} = (2^{\aleph_0})^+ = \Theta$. In particular, $\delta_{\mathcal{S}(\omega_2)} \geq \omega_2$.
- (c) *There are models M of ZFC in which 2^{\aleph_0} is as large as desired and $\delta_{\mathcal{S}(\kappa)} = \kappa^+$ for all infinite $\kappa \leq 2^{\aleph_0}$.*

PROOF. (a) This directly follows from the Kunen-Martin's theorem [Mos09, Theorem 2G.2], which holds in ZF.

(b) $\delta_{\mathcal{S}(\omega)} = \omega_1$ is trivial, so let us show that $\delta_{\mathcal{S}(\omega_1)} = \omega_2$. It is enough to show that for every $\omega_1 \leq \alpha < \omega_2$ there is a $\Delta_{\mathcal{S}(\omega_1)}$ prewellordering \preceq of length α of $\text{LO} \subseteq {}^\omega \times {}^\omega 2$, the set of codes for linear orderings of ω [Kec95, Section 27.C]. For every $\omega \leq \beta < \omega_1$, let WO_β be the set of codes for well-orders on ω of length β , and let NWO be the set of codes for non-well-founded linear orderings of ω . Notice that LO is the disjoint union of the WO_β 's and NWO , which are all Σ_1^1 sets. Given α as above, let $i: \alpha \rightarrow \omega_1 \setminus \omega$ be a bijection and for every $x, y \in \text{LO}$ set

$$x \preceq y \Leftrightarrow x \in \text{NWO} \vee \exists \beta \leq \gamma < \alpha (x \in \text{WO}_{i(\beta)} \wedge y \in \text{WO}_{i(\gamma)}).$$

The relation \preceq is a prewellordering of LO of length α . Using that $\text{WO}_\beta, \text{NWO} \in \Sigma_1^1 \subseteq \mathcal{S}(\omega_1)$ and that under AC the pointclass $\mathcal{S}(\omega_1)$ is closed under unions of size ω_1 (Remark 9.8), it follows that \preceq is $\Delta_{\mathcal{S}(\omega_1)}$, as required.

Finally, to show $\delta_{\mathcal{S}(2^{\aleph_0})} = (2^{\aleph_0})^+$, fix $2^{\aleph_0} \leq \alpha < (2^{\aleph_0})^+$ and let $i: {}^\omega 2 \rightarrow \alpha$ be a bijection: then the relation

$$x \preceq y \Leftrightarrow i(x) \leq i(y)$$

is a prewellordering of ${}^\omega 2$ of length α , and is in $\Delta_{\mathcal{S}(2^{\aleph_0})}$ because every subset of ${}^\omega 2 \times {}^\omega 2$ is trivially 2^{\aleph_0} -Souslin.

(c) By part (b) it is enough to consider the case $\omega_1 \leq \kappa < 2^{\aleph_0}$. By [Har77, Theorem B], there is a model M of ZFC in which 2^{\aleph_0} is as large as desired and every set $A \subseteq {}^\omega 2$ of size $< 2^{\aleph_0}$ is Π_2^1 : we claim that M is as required. Let $A \subseteq {}^\omega 2$ be a set of size κ . Fix $\kappa \leq \alpha < \kappa^+$ and a bijection $i: A \rightarrow \alpha$. Then the binary relation \preceq on ${}^\omega 2$ defined by

$$x \preceq y \Leftrightarrow x \notin A \vee (x, y \in A \wedge i(x) \leq i(y))$$

is a prewellordering of ${}^\omega 2$ of length α . Since $\mathcal{S}(\kappa)$ is closed under unions of size κ , to show that \preceq is $\Delta_{\mathcal{S}(\kappa)}$ it is enough to check that ${}^\omega 2 \setminus A \in \mathcal{S}(\kappa)$. But this follows from the fact that since $|A| < 2^{\aleph_0}$, our choice of M ensures ${}^\omega 2 \setminus A \in \Sigma_2^1 \subseteq \mathcal{S}(\omega_1) \subseteq \mathcal{S}(\kappa)$. \square

REMARK 9.17. Although it is always the case that³⁰ $\check{\mathcal{S}}(\omega) \subseteq \mathcal{S}(\omega_1)$ and $\mathcal{S}(\kappa) \subseteq \mathcal{S}(\kappa')$ for all infinite $\kappa \leq \kappa' \leq 2^{\aleph_0}$, there are models of ZFC in which 2^{\aleph_0} is as large as desired but for every cardinal κ such that $\kappa^+ < 2^{\aleph_0}$ there is $A \subseteq {}^\omega 2$ in $\mathcal{S}(\omega_1)$ such that ${}^\omega 2 \setminus A \notin \mathcal{S}(\kappa)$, so that in general $\check{\mathcal{S}}(\kappa) \not\subseteq \mathcal{S}(\kappa^+)$ when $\kappa > \omega$. To see this let M be a model of ZFC as in the proof of Proposition 9.16(c), fix κ as above, and let $A \subseteq {}^\omega 2$ be such that $|{}^\omega 2 \setminus A| = \kappa^+$. Then $A \in \Sigma_2^1 \subseteq \mathcal{S}(\omega_1)$ by our choice of M , but the set ${}^\omega 2 \setminus A$ does not belong to $\mathcal{S}(\kappa)$ because it cannot satisfy the κ -PSP by $\kappa < |{}^\omega 2 \setminus A| < 2^{\aleph_0}$ (see Proposition 9.5).

Assuming $\forall x \in {}^\omega \omega$ ($x^\#$ exists), the assumption $\delta_2^1 = \omega_2$ has high consistency strength. The next result shows that this is not the case if we drop the existence of sharps.

PROPOSITION 9.18 (AC). *Assume $\text{MA} + \neg\text{CH} + \exists a \in {}^\omega \omega$ ($\omega_1^{L[a]} = \omega_1$). Then $\mathcal{S}(\omega_1) = \Sigma_2^1$, and hence $\delta_2^1 = \omega_2$ by Proposition 9.16(b).*

PROOF. As already observed, $\Sigma_2^1 \subseteq \mathcal{S}(\omega_1)$. For the converse, recall from Remark 9.8 that each $A \in \mathcal{S}(\omega_1)$ is a union of \aleph_1 -many Borel sets. Since Martin and Solovay showed in [MS70] that under the assumptions above every union of at most \aleph_1 -many Borel sets is in Σ_2^1 , we get $A \in \Sigma_2^1$ and we are done. \square

Let us now turn our attention to $\mathcal{S}(\kappa)$ -in-the-codes functions in models with choice. By Proposition 9.16(b) and Remark 5.4(ii), in models of ZFC it always makes sense to speak of $\mathcal{S}(\kappa)$ -in-the-codes functions $f: {}^\omega 2 \rightarrow {}^\kappa 2$ (see Definition 5.3) for $\kappa = \omega$, $\kappa = \omega_1$, or $\kappa = 2^{\aleph_0}$, but in some models this could well be the case for all $\kappa \leq 2^{\aleph_0}$, see Proposition 9.16(c). Notice also that it always makes sense to speak of Σ_2^1 -in-the-codes (and hence $\mathcal{S}(\omega_1)$ -in-the-codes) functions $f: {}^\omega 2 \rightarrow {}^{\omega_1} 2$ even when working in ZF + DC. This is because $\delta_{\mathcal{S}(\omega_1)} \geq \delta_2^1 \geq \omega_1$ (by $\mathcal{S}(\omega) = \Sigma_1^1 \subseteq \Sigma_2^1 \subseteq \mathcal{S}(\omega_1)$), and the norm $\rho: \text{WO} = \bigcup_{\omega \leq \beta < \omega_1} \text{WO}_\beta \rightarrow \omega_1$ sending $x \in \text{WO}$ to the unique $\alpha < \omega_1$ such that $x \in \text{WO}_{\omega+\alpha}$ is easily seen to be a Σ_2^1 -norm on the set $\text{WO} \in \Pi_1^1 \subseteq \Sigma_2^1 \subseteq \mathcal{S}(\omega_1)$. Notice that the definition of $\mathcal{S}(\kappa)$ -in-the-codes functions does not depend on the particular choice of the $\mathcal{S}(\kappa)$ -norm ρ by Remark 5.11(ii). In fact, in models of AC the pointclass $\mathcal{S}(\kappa)$ is closed under projections and finite unions by Lemma 9.7, and it is closed under well-ordered unions of length κ by Remark 9.8: thus it always satisfies the hypothesis of Lemma 5.8 as soon as there is an $\mathcal{S}(\kappa)$ -code for κ (i.e. as soon as it makes sense to speak of $\mathcal{S}(\kappa)$ -in-the-codes functions). Using these facts, we can reformulate Lemma 5.8 as follows.

³⁰In fact, $\check{\mathcal{S}}(\omega) = \Pi_1^1 \subseteq \Sigma_2^1 \subseteq \mathcal{S}(\omega_1)$.

PROPOSITION 9.19 (AC). *Let $\kappa \leq 2^{\aleph_0}$ be such that there is an $\mathbf{S}(\kappa)$ -code for κ . For every $f: {}^\omega 2 \rightarrow {}^\kappa 2$ the following are equivalent:*

- (a) *f is $\mathbf{S}(\kappa)$ -in-the-codes;*
- (b) *$f^{-1}(\tilde{N}_{\alpha,i}) \in \Delta_{\mathbf{S}(\kappa)}$ for every $\alpha < \kappa$ and $i = 0, 1$;*
- (c) *$f^{-1}(U) \in \Delta_{\mathbf{S}(\kappa)}$ for every $U \in \mathcal{B}_p({}^\kappa 2)$.*

This in particular shows that in models of ZFC the notion of an $\mathbf{S}(\kappa)$ -in-the-codes function is nontrivial as soon as $\Delta_{\mathbf{S}(\kappa)} \neq \mathcal{P}({}^\omega 2)$: indeed, by Proposition 9.19 if $A \subseteq {}^\omega 2$ does not belong to $\Delta_{\mathbf{S}(\kappa)}$ and y_0, y_1 are distinct points of ${}^\kappa 2$, the function $f: {}^\omega 2 \rightarrow {}^\kappa 2$ sending all points of A to y_0 and all points of ${}^\omega 2 \setminus A$ to y_1 is not $\mathbf{S}(\kappa)$ -in-the-codes. By Proposition 9.13, this applies to all $\kappa < 2^{\aleph_0}$. Moreover, using Remark 9.8 for small cardinals κ we further have:

COROLLARY 9.20 (AC). *Let $0 \neq n \in \omega$ be such that there is an $\mathbf{S}(\omega_n)$ -code for ω_n . For every $f: {}^\omega 2 \rightarrow {}^{\omega_n} 2$ the following are equivalent:*

- (a) *f is $\mathbf{S}(\omega_n)$ -in-the-codes;*
- (b) *the set $f^{-1}(\tilde{N}_{\alpha,i})$ is the union of \aleph_n -many Borel sets, for every $\alpha < \kappa$ and $i = 0, 1$.*

Finally, using Proposition 9.18 and Corollary 9.20 we also get the following corollary characterizing Σ_2^1 -in-the-codes functions in certain models of ZFC.

COROLLARY 9.21 (AC). *Assume $\mathbf{MA} + \neg \mathbf{CH} + \exists a \in {}^\omega \omega (\omega_1^{L[a]} = \omega_1)$. Then a function $f: {}^\omega 2 \rightarrow {}^{\omega_1} 2$ is Σ_2^1 -in-the-codes if and only if for every $\alpha < \omega_1$ and $i = 0, 1$ the set $f^{-1}(\tilde{N}_{\alpha,i})$ is the union of \aleph_1 -many Borel sets.*

Note that in Proposition 9.19 and in Corollaries 9.20 and 9.21 above we can replace ${}^\kappa 2$ with any space of type κ — see the discussion after Definition 7.9.

9.4. Souslin sets and cardinals in models of determinacy. The structure of the pointclasses $\mathbf{S}(\kappa)$ had been extensively studied under $\mathbf{AD} + \mathbf{DC}$. Here we summarize the most important results, referring the reader to [Jac08, Jac10] for proofs and further results.

LEMMA 9.22 (AD). *If $\kappa < \Theta$ then $\mathbf{S}(\kappa) \neq \mathcal{P}({}^\omega 2)$.*

PROOF. By Remark 9.3, $\text{Tr}(2 \times \kappa) \rightarrow \mathcal{P}(\kappa)$, and therefore $\mathcal{P}(\kappa) \twoheadrightarrow \mathbf{S}(\kappa)$. By Theorem 4.4 ${}^\omega 2 \twoheadrightarrow \mathcal{P}(\kappa)$, so the result follows from Cantor's theorem that ${}^\omega 2$ does not surject onto $\mathcal{P}({}^\omega 2)$. \square

Recall that $\Xi \leq \Theta$ by Lemma 9.12. By Corollary 9.15, under AC we have that $\Xi = 2^{\aleph_0} < (2^{\aleph_0})^+ = \Theta$, so the inequality between Ξ and Θ is strict, but in the context of AD the situation is different as both the cases $\Xi = \Theta$ and $\Xi < \Theta$ can occur.

PROPOSITION 9.23 (AD + DC). *The following are equivalent:*

- (a) $\Xi = \Theta$ (i.e. Souslin cardinals are unbounded below Θ);
- (b) every set of reals is κ -Souslin for some cardinal κ (i.e. $\mathbf{S}(\infty) = \mathbf{S}(\Xi) = \mathcal{P}({}^\omega 2)$);
- (c) Unif (see Section 2.2);
- (d) $\mathbf{AD}_{\mathbb{R}}$.

PROOF. For the equivalence of (b), (c) and (d) see [Ket11, Corollary 5.12]. The implication (a) \Rightarrow (b) is [Woo10, Remark 9.21]. For $\neg(a) \Rightarrow \neg(b)$ apply Lemma 9.22 with $\kappa = \Xi$. \square

On the other hand, if $\Xi < \Theta$ then by Lemma 9.22 the hereditary boldface pointclass $\mathbf{S}(\infty) = \mathbf{S}(\Xi)$ is proper, but it can happen that Ξ is a Souslin cardinal or not. For example, $\mathbf{AD} + \mathbf{V} = \mathbf{L}(\mathbb{R})$ implies that $\Xi = \delta_1^2$ and that δ_1^2 is a Souslin cardinal — in fact it is the δ_1^2 -th Souslin cardinal by (9.1) below. More generally, by results of Steel and Woodin $\mathbf{AD} + \mathbf{DC}(\mathbb{R})$ implies that the Souslin cardinals are closed below Ξ (see [Ket11]). Woodin has isolated the following natural strengthening of AD.

DEFINITION 9.24. AD^+ is the conjunction of the following statements:

- $\text{DC}(\mathbb{R})$,
- $\mathbf{B}_\infty^{(\omega 2)} = \mathcal{P}^{(\omega 2)}$,
- *Ordinal Determinacy*: if $\lambda < \Theta$ and $\pi: {}^\omega \lambda \rightarrow {}^\omega \omega$ is a continuous surjection, then $\pi^{-1}(A)$ is determined, for all $A \subseteq {}^\omega \omega$.

The axiom AD^+ is equivalent (under $\text{AD} + \text{DC}(\mathbb{R})$) to the fact that Souslin cardinals are closed below Θ (see [Ket11, Theorem 7.2] for a proof). Thus assuming AD^+ ,

$$\Xi < \Theta \Leftrightarrow \Xi \text{ is a Souslin cardinal.}$$

Both $\text{AD}_\mathbb{R} + \text{DC}$ and $\text{AD} + \text{V} = \text{L}(\mathbb{R})$ imply AD^+ . Every model of AD known to date does satisfy AD^+ , but it is open whether $\text{AD} \Rightarrow \text{AD}^+$. It is known that the theory $\text{AD} + \neg \text{AD}^+$ has very high consistency strength, if consistent at all (see [KW10, Section 8]).

If Ξ is a Souslin cardinal, then Souslin cardinals are unbounded below Ξ (a fact which is trivially true if Ξ is not a Souslin cardinal), Ξ is regular, and $\mathcal{S}(\Xi)$ is closed also under coprojections (equivalently, the dual pointclass $\check{\mathcal{S}}(\Xi)$ of $\mathcal{S}(\Xi)$ is closed under projections). In particular,

$$(9.1) \quad \Xi \text{ is the } \Xi\text{-th Souslin cardinal.}$$

Since Souslin cardinals are closed below Ξ they are also closed below *every* Souslin cardinal κ , and hence by a result of Kechris $\mathcal{S}(\kappa)$ is always a nonselfdual pointclass for such κ 's; thus $\mathcal{S}(\kappa)$ can be selfdual only if $\kappa = \Xi$ and Ξ is not a Souslin cardinal.

A particular (and important) kind of Souslin cardinals are the ones related to the projective hierarchy:³¹ assuming $\text{AD} + \text{DC}$ the Souslin cardinals below $\delta_\omega := \sup_{n \in \omega} \delta_n^1 = \aleph_{\varepsilon_0}$ are exactly the λ_{2n+1}^1 's and the δ_{2n+1}^1 's, and we have that $\mathcal{S}(\lambda_{2n+1}^1) = \Sigma_{2n+1}^1$ and $\mathcal{S}(\delta_{2n+1}^1) = \Sigma_{2n+2}^1$ [Jac10, Theorem 2.18], and if κ is one of these Souslin cardinals, then $\delta_{\mathcal{S}(\kappa)} = \kappa^+$ (see Section 4.2). This is generalized by the following Proposition 9.25 (compare it with Proposition 9.16, where an analogous result is proved under AC). Recall from [Jac10] that if κ is a limit of Souslin cardinals and has uncountable cofinality, then we can associate to it a canonical Steel pointclass Γ_0 (whose existence is granted by the analysis in [Ste12]), namely: Γ_0 is a nonselfdual boldface pointclass closed under coprojections and such that $\Delta_{\Gamma_0} = \bigcup_{\lambda < \kappa} \mathcal{S}(\lambda)$. By [Jac10, Lemma 3.8], $\mathcal{S}(\kappa)$ is the closure under projections of Γ_0 , and both $\mathcal{S}(\kappa)$ and Γ_0 have the scale property (and hence also the prewellordering property). The proof of the mentioned [Jac10, Lemma 3.8] further shows that $\kappa = \delta_{\Gamma_0}$.

PROPOSITION 9.25 ($\text{AD} + \text{DC}$). *Let κ be a Souslin cardinal.*

- (a) *Either $\delta_{\mathcal{S}(\kappa)} = \kappa$ or $\delta_{\mathcal{S}(\kappa)} = \kappa^+$.*
- (b) *There is an $\mathcal{S}(\kappa)$ -code for κ .*
- (c) *The following are equivalent:*
 - (1) $\delta_{\mathcal{S}(\kappa)} = \kappa$;
 - (2) $\mathcal{S}(\kappa)$ is closed under coprojections;
 - (3) *either $\kappa = \Xi$, or else $\kappa < \Xi$ is a limit of uncountable cofinality of Souslin cardinals falling in Case III of [Jac10, Theorem 3.28], i.e. such that its associated Steel pointclass Γ_0 is closed under projections;*
 - (4) κ is a regular limit of Souslin cardinals and its associated Steel pointclass Γ_0 is closed under projections;

³¹As explained in [Jac10, Theorem 3.28], the “pattern” of Souslin cardinals we are going to describe can be somehow reproduced above any $\kappa < \Xi$ which is a limit of Souslin cardinals. We repeatedly use this fact in the proof of Proposition 9.25.

(5) κ is a regular limit of Souslin cardinals and $\mathbf{S}(\kappa)$ coincides with its associated Steel pointclass $\mathbf{\Gamma}_0$.

In particular, $\delta_{\mathbf{S}(\kappa)} = \kappa^+$ whenever κ is not a regular limit of Souslin cardinals.

PROOF. (a) By [Jac10, Theorem 2.18] (see the paragraph preceding this proposition), we can assume without loss of generality that κ is above a limit of Souslin cardinals, so that we can use the deep and remarkable classification of the Souslin cardinals presented e.g. in [Jac10, Theorem 3.28]. The inequality $\delta_{\mathbf{S}(\kappa)} \leq \kappa^+$ immediately follows from the Kunen-Martin's theorem (see e.g. [Mos09, Theorem 2G.2]), so that we just need to show $\kappa \leq \delta_{\mathbf{S}(\kappa)}$. From [Mos09, Theorem 7D.8] we have that $\delta_{\mathbf{S}(\kappa)}$ is a cardinal, and that it is of uncountable cofinality by closure of $\mathbf{S}(\kappa)$ under countable unions (Lemma 9.7).

Assume first that κ is not a limit of Souslin cardinals itself (so that, in particular, there is a largest $\kappa' < \kappa$ which is a limit of Souslin cardinals, which can be taken as basis to carry out the analysis provided in [Jac10, Theorem 3.28]). We consider two distinct cases, depending on whether κ has countable cofinality or not. In the former case, it follows from [Jac10, Theorem 3.28] that $\kappa = \lambda_{2i+1}$ for some $i \in \omega$ and that $\dot{\mathbf{S}}(\kappa) = \mathbf{\Pi}_{2i+1}$ has the scale property. Since the (regular) $\dot{\mathbf{S}}(\kappa)$ -norms constituting a $\dot{\mathbf{S}}(\kappa)$ -scale on an arbitrary set $A \subseteq {}^\omega\omega$ in $\dot{\mathbf{S}}(\kappa)$ are into $\delta_{\mathbf{S}(\kappa)}$, if $\delta_{\mathbf{S}(\kappa)} \leq \kappa$ then the mentioned scale would canonically yield a κ -Souslin representation of A , and therefore we would get $\dot{\mathbf{S}}(\kappa) \subseteq \mathbf{S}(\kappa)$: but this contradicts the fact that $\mathbf{S}(\kappa)$ is nonselfdual. Thus we get $\kappa < \delta_{\mathbf{S}(\kappa)}$, and thus also $\delta_{\mathbf{S}(\kappa)} = \kappa^+$ because $\delta_{\mathbf{S}(\kappa)}$ is a cardinal. If instead κ has uncountable cofinality, then $\mathbf{S}(\kappa)$ has the scale property by [Jac10, Lemma 3.7]. Let $A \subseteq {}^\omega\omega$ be such that $A \in \mathbf{S}(\kappa) \setminus \dot{\mathbf{S}}(\kappa)$, and consider an arbitrary $\mathbf{S}(\kappa)$ -scale $\langle \rho_n \mid n \in \omega \rangle$ on it. Arguing as in the proof of Lemma 9.12, by our choice of A we get

$$\sup \{ \alpha \in \text{Ord} \mid \exists n \in \omega \exists x \in A (\rho_n(x) = \alpha) \} = \kappa.$$

Moreover, since κ has uncountable cofinality there is $\bar{n} \in \omega$ such that the $\mathbf{S}(\kappa)$ -norm $\rho_{\bar{n}}$ on A is onto κ . Then for every $\alpha < \kappa$ we can construct a $\Delta_{\mathbf{S}(\kappa)}$ prewellordering \preceq of ${}^\omega\omega$ of length $\alpha + 2$ by fixing $z \in A$ such that $\rho_{\bar{n}}(z) = \alpha$ and then setting

$$x \preceq y \Leftrightarrow y \notin A' \vee (x \in A' \wedge y \in A' \wedge \rho_{\bar{n}}(x) \leq \rho_{\bar{n}}(y)),$$

where $A' = \{x \in {}^\omega\omega \mid x \in A \wedge \rho_{\bar{n}}(x) \leq \rho_{\bar{n}}(z)\} \in \Delta_{\mathbf{S}(\kappa)}$. Therefore $\kappa \leq \delta_{\mathbf{S}(\kappa)}$ again.

Assume now that κ is a limit of Souslin cardinals. Then for every $\alpha < \kappa$ there is $\alpha < \lambda < \kappa$ such that λ is a Souslin cardinal and is not a limit of Souslin cardinals, so that $\delta_{\mathbf{S}(\lambda)} \geq \lambda$ by the above proof. Since $\lambda \leq \kappa$ implies $\delta_{\mathbf{S}(\kappa)} \geq \delta_{\mathbf{S}(\lambda)}$, we get $\delta_{\mathbf{S}(\kappa)} > \alpha$, whence $\kappa \leq \delta_{\mathbf{S}(\kappa)}$.

(b) If $\kappa < \delta_{\mathbf{S}(\kappa)}$ then we are done by Remark 5.4(ii), so let us assume $\kappa = \delta_{\mathbf{S}(\kappa)}$. Notice that this implies that κ is of uncountable cofinality because, as recalled at the beginning of the proof of part (a), $\text{cof}(\delta_{\mathbf{S}(\kappa)}) > \omega$. By³² [Jac10, Lemmas 3.7 and 3.8], we then get that $\mathbf{S}(\kappa)$ has the scale property. Let $A \subseteq {}^\omega\omega$ be in $\mathbf{S}(\kappa) \setminus \bigcup_{\lambda < \kappa} \mathbf{S}(\lambda)$, and let $\langle \rho_n \mid n \in \omega \rangle$ be an $\mathbf{S}(\kappa)$ -scale on A . Arguing as in the proof of Lemma 9.12 and using again the fact that κ has uncountable cofinality, by the choice of A there is at least one $\bar{n} \in \omega$ such that $\rho_{\bar{n}}$ has length κ : setting $\rho := \rho_{\bar{n}}$ we get the desired result.

(c) First we prove the equivalence between the conditions (2)–(5). Assume (2), so that in particular $\kappa > \omega$. Then κ is a regular limit of Souslin cardinals by [Jac10, Lemma 3.6]. As recalled before this proposition, in this situation $\mathbf{S}(\kappa)$ is the closure under projection of its associated Steel pointclass $\mathbf{\Gamma}_0$, so that in particular $\mathbf{S}(\kappa) \supseteq \mathbf{\Gamma}_0$. If this inclusion were proper, then $\mathbf{\Gamma}_0$ could not be closed under projections, and thus κ would fall in Case II of [Jac10,

³²In fact we show in part (c) that if $\kappa = \delta_{\mathbf{S}(\kappa)}$ then κ is a regular limit of Souslin cardinals, so only [Jac10, Lemma 3.8] needs to be applied in this case. However, since the proof of (c) partially relies on (b), here we mentioned also [Jac10, Lemma 3.7] to make it evident that there is no circularity in the argument.

Theorem 3.28]: but this would contradict the closure under coprojections of $\mathbf{S}(\kappa)$. It follows that $\mathbf{S}(\kappa) = \mathbf{\Gamma}_0$, i.e. (5). The implications from (5) to (4) and from (4) to (3) are obvious (recall that under our assumptions $\mathbf{S}(\kappa)$ is always closed under projections by Lemma 9.7), so let us show that (3) implies (2). If $\kappa = \Xi$ then the result follows from [Jac08, Lemma 2.20]. In the remaining case, $\mathbf{S}(\kappa) = \mathbf{\Gamma}_0$ because $\mathbf{S}(\kappa)$ is the closure under projections of $\mathbf{\Gamma}_0$, and thus (2) follows from the fact that $\mathbf{\Gamma}_0$ is closed under coprojections.

Since (5) easily implies (1) (because $\kappa = \delta_{\mathbf{\Gamma}_0}$), to conclude our proof it is enough to show that (1) implies (4).

CLAIM 9.25.1. *If $\kappa = \delta_{\mathbf{S}(\kappa)}$, then κ is a regular limit of Souslin cardinals.*

PROOF OF THE CLAIM. We prove the contrapositive, i.e. that if κ is *not* a regular limit of Souslin cardinals then $\delta_{\mathbf{S}(\kappa)} \neq \kappa$ (whence $\delta_{\mathbf{S}(\kappa)} = \kappa^+$ by (a)). This is trivial for Souslin cardinals κ of countable cofinality: as observed in the proof of part (b), in such case $\delta_{\mathbf{S}(\kappa)} \neq \kappa$ because $\delta_{\mathbf{S}(\kappa)}$ has uncountable cofinality by closure of $\mathbf{S}(\kappa)$ under countable unions. Now assume that κ has uncountable cofinality and is not a limit of Souslin cardinals, and let λ be the largest Souslin cardinal smaller than κ . By (the proof of) [Jac10, Lemma 3.7], λ is of countable cofinality, $\kappa = \lambda^+$, and $\check{\mathbf{S}}(\lambda)$ has the prewellordering property. Since $\check{\mathbf{S}}(\lambda)$, being nonselfdual, admits a universal set and is closed under coprojections, we get from e.g. [Jac08, Lemma 1.3] that for every $A \subseteq {}^\omega\omega$ such that $A \in \check{\mathbf{S}}(\lambda) \setminus \mathbf{S}(\lambda)$ there is an $\check{\mathbf{S}}(\lambda)$ -norm ρ on A of length $\delta_{\mathbf{S}(\lambda)}$. Since κ is a Souslin cardinal, $\mathbf{S}(\kappa)$ is nonselfdual and $\mathbf{S}(\lambda) \subset \mathbf{S}(\kappa)$, whence $\mathbf{S}(\lambda) \cup \check{\mathbf{S}}(\lambda) \subseteq \Delta_{\mathbf{S}(\kappa)}$. Therefore, the prewellordering \preceq of ${}^\omega\omega$ defined by

$$(9.2) \quad x \preceq y \Leftrightarrow y \notin A \vee (x \in A \wedge y \in A \wedge \rho(x) \leq \rho(y))$$

is in $\Delta_{\mathbf{S}(\kappa)}$ and has length $\delta_{\mathbf{S}(\lambda)} + 1$. Since $\delta_{\mathbf{S}(\lambda)} = \lambda^+$ (because λ , being a Souslin cardinal of countable cofinality, falls in the already considered trivial case), this implies $\delta_{\mathbf{S}(\kappa)} > \lambda^+ = \kappa$.

Finally, let us assume that κ is a limit of Souslin cardinals such that $\omega < \text{cof}(\kappa) < \kappa$, and fix a cofinal map $f: \text{cof}(\kappa) \rightarrow \kappa$. By part (b), for some κ -Souslin $A \subseteq {}^\omega\omega$ there is a $\mathbf{S}(\kappa)$ -norm ρ on A of length κ . Fix a Souslin cardinal $\text{cof}(\kappa) < \lambda < \kappa$ and some $\Delta_{\mathbf{S}(\lambda)}$ -norm $\sigma: {}^\omega\omega \rightarrow \text{cof}(\kappa)$ (which exists because by part (a) we have $\text{cof}(\kappa) < \lambda \leq \delta_{\mathbf{S}(\lambda)}$), so that the strict well-founded relation on ${}^\omega\omega$ associated to σ is in $\mathbf{S}(\lambda)$. Then apply the first Coding Lemma [Jac10, Theorem 2.12] to

$$R := \{(z, w) \in {}^\omega\omega \times {}^\omega\omega \mid w \in A \wedge \rho(w) = f(\sigma(z))\}$$

to obtain an $\mathbf{S}(\lambda)$ -set $C \subseteq {}^\omega\omega \times {}^\omega\omega$ such that

- for every $\alpha < \text{cof}(\kappa)$ there is $(z, w) \in C$ with $\sigma(z) = \alpha$, and
- for every $(z, w) \in C$, $w \in A$ and $\rho(w) = f(\sigma(z))$.

Then the prewellordering \preceq of ${}^\omega\omega \times {}^\omega\omega$ defined by setting $(x, y) \preceq (x', y')$ if and only if

$$\begin{aligned} \sigma(x) < \sigma(x') \vee [\sigma(x) = \sigma(x') \wedge (\exists (z, w) \in C (\sigma(z) = \sigma(x) \wedge \rho(y') > \rho(w)) \vee \\ \exists (z, w) \in C (\sigma(z) = \sigma(x) \wedge \rho(y) \leq \rho(y') \leq \rho(w)))] \end{aligned}$$

has length $\sum_{\alpha < \text{cof}(\kappa)} (f(\alpha) + 2) = \kappa$. Moreover, by our choice of C we also have that $(x, y) \preceq (x', y')$ if and only if

$$\begin{aligned} \sigma(x) < \sigma(x') \vee [\sigma(x) = \sigma(x') \wedge (\forall (z, w) \in C (\sigma(z) = \sigma(x) \Rightarrow \rho(y') > \rho(w)) \vee \\ \forall (z, w) \in C (\sigma(z) = \sigma(x) \Rightarrow \rho(y) \leq \rho(y') \leq \rho(w)))] \end{aligned}$$

and hence \preceq is in $\Delta_{\mathbf{S}(\kappa)}$ (here we are using the fact that $\mathbf{S}(\lambda) \cup \check{\mathbf{S}}(\lambda) \subseteq \Delta_{\mathbf{S}(\kappa)}$). Since ${}^\omega\omega$ and ${}^\omega\omega \times {}^\omega\omega$ are homeomorphic, this shows that $\delta_{\mathbf{S}(\kappa)} > \kappa$, so we are done. \square

Now assume (1), so that κ is a regular limit of Souslin cardinals by the above Claim 9.25.1. As observed in [Jac08], from this and [Ste12, Theorem 2.1] it follows that Γ_0 is closed under finite unions. Since Γ_0 is also closed under coprojections and admits a universal set (being nonselfdual), by the prewellordering property for Γ_0 and e.g. [Jac08, Lemma 1.3] we get that for any $A \in \Gamma_0 \setminus \check{\Gamma}_0$ there is a Γ_0 -norm ρ on A of length $\delta_{\Gamma_0} = \kappa$. Assume towards a contradiction that Γ_0 is not closed under projections. Then $\mathcal{S}(\kappa)$, which is the closure under projections of Γ_0 , would contain both Γ_0 and its dual $\check{\Gamma}_0$, so that $\Gamma_0 \cup \check{\Gamma}_0 \subseteq \Delta_{\mathcal{S}(\kappa)}$. But then the prewellordering \preceq obtained from the Γ_0 -norm ρ as in (9.2) would be in $\Delta_{\mathcal{S}(\kappa)}$: since its length is $\kappa+1$, this would contradict our assumption $\kappa = \delta_{\mathcal{S}(\kappa)}$. Thus Γ_0 is closed under projections and (4) holds. \square

REMARK 9.26. We currently do not know whether there are regular limits of Souslin cardinals which may actually fall into Case II of [Jac10, Theorem 3.28], i.e. whether all regular limits of Souslin cardinals need to satisfy the equivalent conditions of Proposition 9.25(c).

As we already observed, Souslin cardinals are always closed and unbounded below Ξ : for our purposes, it is important to notice that the analysis of [Jac10] shows that also the *regular* Souslin cardinals are unbounded below Ξ .

LEMMA 9.27 (AD + DC). *Regular Souslin cardinals are unbounded below Ξ . In particular, for every Souslin cardinal κ there is a regular Souslin cardinal $\kappa' \geq \kappa$.*

PROOF. Given $\alpha < \Xi$, let $\bar{\kappa}$ be the smallest Souslin cardinal above α and κ be the largest limit of Souslin cardinals $\leq \bar{\kappa}$ (such $\bar{\kappa}$ and κ exists because, as already recalled in the discussion preceding this lemma, there are club-many Souslin cardinals below Ξ). Applying [Jac10, Theorem 3.28] to such κ , we obtain that there is $i \in \omega$ such that $\bar{\kappa} \leq \delta_{2i+1}$, where $\delta_{2i+1} = (\lambda_{2i+1})^+$ is as in any of Case I–III of [Jac10, Theorem 3.28]. Since in all these cases

$$\delta_{2i+1} = \delta_{\Pi_{2i+1}} = \delta_{\Sigma_{2i+1}} = \delta_{\mathcal{S}(\lambda_{2i+1})},$$

by the Kunen-Martin's theorem (see e.g. [Mos09, Theorem 2G.2]) the Souslin cardinal δ_{2i+1} is the supremum of the lengths of the λ_{2i+1} -Souslin strict well-founded relations on ${}^\omega\omega$, and thus it is a regular cardinal by [Jac10, Lemma 2.16]. Therefore since $\alpha < \bar{\kappa} \leq \delta_{2i+1} < \Xi$ we are done. The second part of the lemma follows from the first one and the fact that if Ξ is a Souslin cardinal then it is also regular. \square

We now consider $\mathcal{S}(\kappa)$ -in-the-codes functions in models of determinacy. By Proposition 9.25(b), if κ is a Souslin cardinal then it always makes sense to speak about $\mathcal{S}(\kappa)$ -in-the-codes functions $f: {}^\omega 2 \rightarrow {}^\kappa 2$. Notice also that the definition of $\mathcal{S}(\kappa)$ -in-the-codes functions does not depend on the particular choice of the $\mathcal{S}(\kappa)$ -norm ρ by Remark 5.11(ii) and Lemma 9.7. Using these facts, we can reformulate Proposition 5.10 as follows. (When $\kappa = \omega$ we can dispense with all determinacy assumptions in Proposition 9.28 and Corollary 9.29.)

PROPOSITION 9.28 (AD + DC). *Let κ be a Souslin cardinal. For every $f: {}^\omega 2 \rightarrow {}^\kappa 2$ the following are equivalent:*

- (a) *f is $\mathcal{S}(\kappa)$ -in-the-codes:*
- (b) *$f^{-1}(\tilde{N}_{\alpha,i}) \in \Delta_{\mathcal{S}(\kappa)}$ for every $\alpha < \kappa$ and $i = 0, 1$;*
- (c) *$f^{-1}(U) \in \Delta_{\mathcal{S}(\kappa)}$ for every $U \in \mathcal{B}_p({}^\kappa 2)$.*

PROOF. The pointclass $\mathcal{S}(\kappa)$ satisfies the hypotheses of Proposition 5.10 by Lemma 9.7. \square

Notice that in Proposition 9.28 (as well as in the subsequent Corollary 9.29) we can replace ${}^\kappa 2$ with any space of type κ — see the discussion after Definition 7.9. Moreover, arguing as in the paragraph after Proposition 9.19 one may observe that the above proposition implies in particular that also in models of AD+DC the notion of an $\mathcal{S}(\kappa)$ -in-the-codes function $f: {}^\omega 2 \rightarrow {}^\kappa 2$

is nontrivial as soon as $\Delta_{S(\kappa)} \neq \mathcal{P}({}^\omega 2)$: by Proposition 9.23, this is the case for all Souslin cardinals κ .

Recall that by Proposition 9.10, every $S(\kappa)$ -in-the-codes function $f: {}^\omega 2 \rightarrow {}^\kappa 2$ is weakly $\kappa + 1$ -Borel. Since when $\kappa = \lambda_{2n+1}^1$ for some $n \in \omega$ we have $S(\kappa) = \Sigma_{2n+1}^1$, the next corollary shows that in certain cases the two notions coincide.

COROLLARY 9.29 (AD + DC). *Let $\kappa = \lambda_{2n+1}^1$ for some $n \in \omega$. Then a function $f: {}^\omega 2 \rightarrow {}^\kappa 2$ is Σ_{2n+1}^1 -in-the-codes if and only if it is weakly $\kappa + 1$ -Borel.*

PROOF. Since $\Sigma_{2n+1}^1 = S(\lambda_{2n+1}^1)$, the forward direction is just an instantiation of Proposition 9.10, while the other direction follows from Corollary 5.7(b) and Proposition 9.28. \square

10. The main construction

In [LR05], the main technical construction for proving the completeness (for analytic quasi-orders) of the relation \sqsubset_{CT}^ω of embeddability between countable combinatorial trees is a map which given an arbitrary $S \in \text{Tr}(2 \times \omega)$ provides a combinatorial tree \mathbb{G}_S . The tree \mathbb{G}_S is constructed in two steps: first a fixed combinatorial tree \mathbb{G}_0 is defined, independent of S , and then certain auxiliary combinatorial trees, called *forks*, coding the tree S are added to \mathbb{G}_0 . In order to prove the invariant universality of the embeddability relation between countable structures, in [FMR11] this construction is improved so that the resulting \mathbb{G}_S is rigid, i.e. without nontrivial automorphisms. As explained in that paper, there are at least two ways to ensure that the resulting structure is rigid:

- (1) enrich the language for graphs $\mathcal{L} = \{\mathbf{E}\}$ with an additional binary relational symbol \trianglelefteq , and then expand \mathbb{G}_S to a so-called *ordered* combinatorial tree $\bar{\mathbb{G}}_S$ by interpreting \trianglelefteq as a well-order on the vertices of \mathbb{G}_S (see the proof of [FMR11, Theorem 2.4]);
- (2) enlarge \mathbb{G}_0 to a rigid combinatorial tree \mathbb{G}_1 , and then add the forks to \mathbb{G}_1 (see the proof of [FMR11, Theorem 2.4] or [CMMR13, Section 3]).

Although the construction in (1) is simpler, such approach is slightly unnatural because it forces us to consider more complicated structures, while one of the motivations in using combinatorial trees in [LR05] was that they are rather simple objects. The approach (2), even if technically more involved, gives instead the stronger result that already \sqsubset_{CT}^ω is invariantly universal, and it has proven to be quite useful in the applications to infinite combinatorics, topology, analysis, and Banach space theory [CMMR13].

In this paper we generalize both constructions (1) and (2) to uncountable cardinals κ . As in the classical case, the approach using combinatorial trees (which is developed in this section and in the subsequent Sections 11–12) is preferable as it deals with more elementary objects, and it yields full generalizations of Theorems 1.1 and 1.4, as well as most of the applications, including results on non-separable (discrete, ultrametric) complete metric spaces and on non-separable Banach spaces. A generalization of Theorem 1.5 seems to require the approach via ordered combinatorial trees and is postponed to Section 13.

Fix an infinite cardinal κ . We adapt the constructions from [LR05, FMR11, CMMR13] to define a map

$$\text{Tr}(2 \times \kappa) \rightarrow \text{CT}_\kappa, \quad S \mapsto \mathbb{G}_S,$$

where CT_κ is the set of (codes for) all combinatorial trees of size κ from (2.5) on page 18. Such map will then be used in Sections 11 and 12 to prove that \sqsubset_{CT}^κ , the embeddability relation on CT_κ , is invariantly universal³³ — and hence also complete — for κ -Souslin quasi-orders. Since as explained in the previous paragraph we want to work just with combinatorial trees (without

³³The notion of invariant universality is given in Definition 12.1.

any additional order on their vertices), we will follow the approach (2) briefly described above, that is:

- construct a basic $\mathbb{G}_0 \in \text{CT}_\kappa$ which is independent of the given input $S \in \text{Tr}(2 \times \kappa)$;
- enlarge \mathbb{G}_0 to a suitable $\mathbb{G}_1 \in \text{CT}_\kappa$ (still independent of S) to get a sufficiently rigid structure;
- to get the final $\mathbb{G}_S \in \text{CT}_\kappa$, add to \mathbb{G}_1 some forks which code enough information on S .

REMARK 10.1. As already observed in [LR05], it is easy to check that we could systematically replace combinatorial trees with *rooted* combinatorial trees in all the constructions and results below — just define the empty sequence \emptyset to be the root of the combinatorial tree \mathbb{G}_0 (and hence also of \mathbb{G}_1 and of every combinatorial tree of the form \mathbb{G}_S for $S \in \text{Tr}(2 \times \kappa)$), and check that all proofs go through.

Notice that as for the basic case $\kappa = \omega$ considered in [LR05, FMR11, CMMR13], the preparatory enlargement from \mathbb{G}_0 to \mathbb{G}_1 is necessary only for the proof of the invariant universality of $\sqsubset_{\text{CT}}^\kappa$: a variant of the main construction in which we attach the forks directly to \mathbb{G}_0 would already enable us to prove the completeness of $\sqsubset_{\text{CT}}^\kappa$ (see Remark 11.9).

Let us first fix some notation concerning combinatorial trees. The language of graphs $\mathcal{L} = \{\mathbf{E}\}$ consists of one binary relational symbol, and each graph $G = (V, E)$ (see Section 2.6.1) is identified with the \mathcal{L} -structure $X = \langle X; \mathbf{E}^X \rangle$ with $X := V$ and $\mathbf{E}^X := \{(v_0, v_1) \in V^2 \mid \{v_0, v_1\} \in E\}$ (so that \mathbf{E}^X is an irreflexive and symmetric relation on X). Recall from page (2.5) that CT_κ is the collection of (codes for) all *combinatorial trees* of the form (κ, E) with $E \subseteq [\kappa]^2$. In fact,

$$\text{CT}_\kappa = \text{Mod}_{\sigma_{\text{CT}}}^\kappa,$$

where $\text{Mod}_{\sigma_{\text{CT}}}^\kappa$ is defined as in (8.3) with σ_{CT} the $\mathcal{L}_{\omega_1\omega}$ -sentence axiomatizing combinatorial trees:

$$\begin{aligned} (\sigma_{\text{CT}}) \quad & \forall v_0 \neg(v_0 \mathbf{E} v_0) \wedge \forall v_0 \forall v_1 (v_0 \mathbf{E} v_1 \Rightarrow v_1 \mathbf{E} v_0) \wedge \\ & \bigwedge_{n \in \omega} \neg \exists \langle v_i \mid i \leq n+2 \rangle \left[\left(\bigwedge_{i < j \leq n+2} v_i \neq v_j \right) \wedge \left(\bigwedge_{i < n+2} v_i \mathbf{E} v_{i+1} \right) \wedge v_0 \mathbf{E} v_{n+2} \right] \wedge \\ & \forall v_0 \forall v_1 \left[\bigvee_{n \in \omega} \exists \langle v_{i+2} \mid i \leq n+1 \rangle \left(v_0 \simeq v_2 \wedge v_1 \simeq v_{n+3} \wedge \bigwedge_{i < n+1} v_{i+2} \mathbf{E} v_{i+3} \right) \right]. \end{aligned}$$

In order to simplify the notation, we will further abbreviate the embeddability and isomorphism relations $\sqsubset_{\sigma_{\text{CT}}}^\kappa$ and $\cong_{\sigma_{\text{CT}}}^\kappa$ on CT_κ (see page 55) with $\sqsubset_{\text{CT}}^\kappa$ and \cong_{CT}^κ , respectively.

10.1. The combinatorial trees \mathbb{G}_0 and \mathbb{G}_1 . We now start our main construction. The **doubling** of a descriptive set-theoretic tree T is the tree T^d obtained by replacing each node s of T different from \emptyset with two nodes $s^- < s^+$ in T^d . Figure 1 shows the doubling of a finite tree. In order to simplify the notation, the nodes s^+ of T^d will simply be denoted by s . Moreover, T^d will be always identified with a combinatorial tree on the set of its nodes.

Applying the doubling procedure to ${}^{<\omega}\kappa$, we obtain the combinatorial tree \mathbb{G}_0 . Formally:

DEFINITION 10.2. \mathbb{G}_0 is the graph on the disjoint union

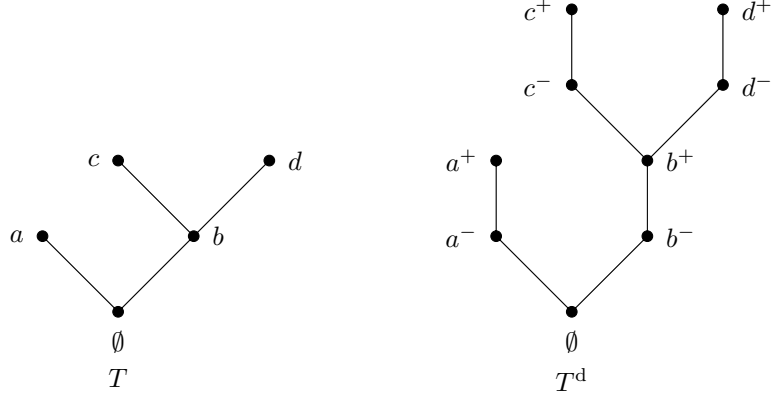
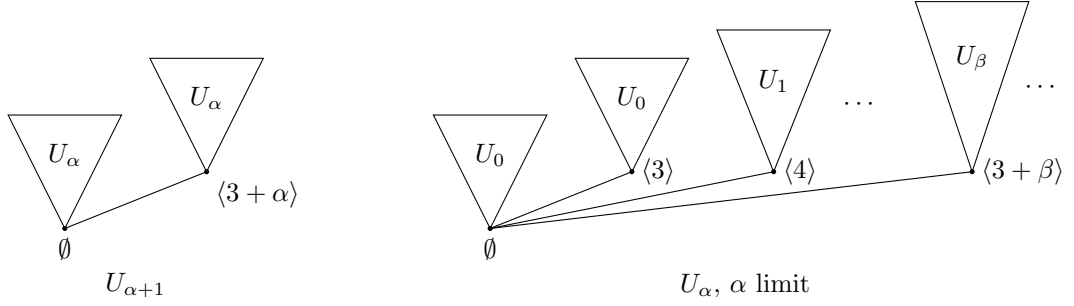
$$G_0 := {}^{<\omega}\kappa \uplus \{s^- \mid \emptyset \neq s \in {}^{<\omega}\kappa\}$$

with edges

$$\mathbf{E}^{\mathbb{G}_0} := \{\{s, s^-\} \mid s \in {}^{<\omega}\kappa \setminus \{\emptyset\}\} \cup \{\{s, (s^\alpha)^-\} \mid s \in {}^{<\omega}\kappa, \alpha < \kappa\}.$$

We next enlarge \mathbb{G}_0 to the new combinatorial tree \mathbb{G}_1 . Given a descriptive set-theoretic tree $U \subseteq {}^{<\omega}\kappa$, the **width** of U is the ordinal

$$\text{w}(U) := \sup \{\gamma + 1 \mid \langle \gamma \rangle \in U\}.$$

FIGURE 1. The doubling of a tree T .FIGURE 2. The trees U_α .

We construct a sequence $\langle U_\alpha \mid \alpha < \kappa \rangle$ of descriptive set-theoretic trees on κ as follows (see Figure 2):

$$\begin{aligned}
 U_0 &= {}^3 3, \\
 U_{\alpha+1} &= U_\alpha \cup \{ \langle w(U_\alpha) \rangle \hat{\smallfrown} s \mid s \in U_\alpha \} \\
 U_\alpha &= \bigcup_{\beta < \alpha} U_\beta \qquad \alpha \text{ limit.}
 \end{aligned}$$

By construction, for every $\alpha < \kappa$

$$(10.1) \quad w(U_\alpha) = 3 + \alpha < \kappa \quad \text{and} \quad U_\alpha \subseteq {}^{<\omega} w(U_\alpha) = {}^{<\omega} (3 + \alpha).$$

Each U_α is identified with the corresponding combinatorial tree (that is with the graph on the nodes of U_α obtained by linking with an edge all pairs of nodes $x, y \in U_\alpha$ such that $x = y^*$, for y^* as in (2.4)). The vertex \emptyset is called the root of U_α .

REMARK 10.3. Each U_α can be construed as a combinatorial tree (with root \emptyset). If $\kappa = \lambda^+$, then using (10.1) we get that $|U_\alpha| \leq \lambda$ for all $\alpha < \kappa$, so each U_α is isomorphic to an element of CT_λ (at least when $\lambda \leq \alpha$). Therefore by choosing a bijection between U_α and λ we could construct the U_α 's so that they belong to CT_λ , when $\alpha \geq \lambda$. The appeal to AC cannot be avoided — if the PSP holds and $\kappa = \omega_1$, then there is no ω_1 -sequence of distinct elements of CT_ω . In our

construction we avoided this issue by relaxing the requirement that a combinatorial tree of size λ have domain equal to λ .

We now define the combinatorial tree \mathbb{G}_1 by connecting a new vertex \hat{s} to each $s \in {}^{<\omega}\kappa \subseteq G_0$ and then appending a copy of $U_{\langle\langle s \rangle\rangle}$ to \hat{s} (where $\langle\langle \cdot \rangle\rangle$ is the function coding sequences by ordinals from (2.2)) by adding an edge between such vertex and the root of $U_{\langle\langle s \rangle\rangle}$. More formally,

DEFINITION 10.4. \mathbb{G}_1 is the combinatorial tree with set of vertices

$$G_1 := G_0 \uplus \{\hat{s} \mid s \in {}^{<\omega}\kappa\} \uplus \{(s, t) \in {}^{<\omega}\kappa \times {}^{<\omega}\kappa \mid t \in U_{\langle\langle s \rangle\rangle}\}$$

and edge relation $\mathbf{E}^{\mathbb{G}_1}$ defined by:

- if $x, y \in G_0$, $x \mathbf{E}^{\mathbb{G}_1} y \Leftrightarrow x \mathbf{E}^{G_0} y$;
- $s \mathbf{E}^{\mathbb{G}_1} \hat{s}$ and $\hat{s} \mathbf{E}^{\mathbb{G}_1} s$ for every $s \in {}^{<\omega}\kappa$;
- $\hat{s} \mathbf{E}^{\mathbb{G}_1} (s, \emptyset)$ and $(s, \emptyset) \mathbf{E}^{\mathbb{G}_1} \hat{s}$ for every $s \in {}^{<\omega}\kappa$;
- for every $s \in {}^{<\omega}\kappa$ and $t, t' \in U_{\langle\langle s \rangle\rangle}$, $(s, t) \mathbf{E}^{\mathbb{G}_1} (s, t') \Leftrightarrow t \mathbf{E}^{U_{\langle\langle s \rangle\rangle}} t'$;
- no other $\mathbf{E}^{\mathbb{G}_1}$ -relation holds.

As explained above, the purpose of moving from \mathbb{G}_0 to \mathbb{G}_1 is to obtain a more rigid structure. In order to prove that \mathbb{G}_1 has few nontrivial automorphisms, we first need to consider some technical properties of the U_α 's which will also be useful in the proof of our invariant universality result (Section 12). Such technical analysis is further complicated by the fact that for our applications we need to work in ZF, and hence we have to ensure that all “choices” needed in the constructions can be done in a canonical way. However these technical details are only needed for the proof of invariantly universality: the reader who is just interested in the completeness result (Section 11) may safely skip the rest of this section and directly jump to Section 10.2.

The relevant properties of the graphs U_α are summarized in the next lemma.

LEMMA 10.5. *Let $\alpha, \beta < \kappa$.*

- (a) *Except for the root \emptyset (which has degree 3 if $\alpha = 0$ and ≥ 4 otherwise), all vertices of U_α are either terminal (i.e. they have a unique neighbor) or have degree ≥ 4 . Moreover, each vertex of U_α has degree $\leq 1 + w(U_\alpha) = 4 + \alpha < \kappa$.*
- (b) *If $\alpha \leq \beta$, then $U_\alpha \subseteq U_\beta$. In particular, the identity map embeds U_α into U_β and fixes the root \emptyset .*
- (c) *For $\beta < \alpha < \kappa$ there is no embedding of U_α into U_β sending \emptyset to itself. In particular, there is an isomorphism between U_α and U_β fixing \emptyset if and only if $\alpha = \beta$.*
- (d) *There is no infinite path through U_α . (Equivalently, U_α is well-founded when construed as a descriptive set-theoretic tree on κ of height $\leq \omega$.)*
- (e) *Let i be an automorphism of U_α . If $\alpha = \gamma + 1$ is a successor ordinal, then either $i(\emptyset) = \emptyset$ or $i(\emptyset) = \langle w(U_\gamma) \rangle$ (and both the possibilities may be realized), while if $\alpha = 0$ or α is limit then $i(\emptyset) = \emptyset$.*
- (f) *Suppose that X is an isomorphic copy of U_α with domain $\subseteq \kappa$. Then there is a canonical isomorphism*

$$(10.2) \quad i_{X, \alpha}: X \rightarrow U_\alpha.$$

PROOF. Parts (a), (b) and (d) are obvious and directly follow from the construction of the U_α 's and (10.1).

A combinatorial tree G with root r and *without infinite paths* can be seen as a well-founded descriptive set-theoretic tree, and hence it has a rank function $\rho_{G, r}: G \rightarrow \text{Ord}$ defined by

$$\rho_{G, r}(a) := \sup \{ \rho_{G, r}(b) + 1 \mid b \neq a \text{ and the unique path joining } r \text{ to } b \text{ passes through } a \}.$$

Thus $\rho_{G,r}(a) = 0$ if and only if a has degree 1 in G , and then set

$$\rho(G, r) := \sup \{ \rho_{G,r}(a) + 1 \mid a \in G, a \neq r \} = \rho_{G,r}(r).$$

Notice that for every two rooted combinatorial trees (G, r) , (G', r') , if $(G, r) \sqsubset (G', r')$ then $\rho(G, r) \leq \rho(G', r')$ (whence in particular $\rho(G, r) = \rho(G', r')$ whenever $(G, r) \cong (G', r')$). Then (c) easily follows from the fact that one can show by induction on α that $\rho(U_\alpha, \emptyset) = 3 + \alpha$.

We now prove (e). Assume first $\alpha = \gamma + 1$, and let $r \in U_\alpha$ be arbitrary. Then either $\rho_{U_\alpha,r}(\emptyset) = \rho_{U_\alpha,\emptyset}(\emptyset)$ (if $\langle w(U_\gamma) \rangle \not\subseteq r$), or else $\rho_{U_\alpha,r}(w(U_\gamma)) = \rho_{U_\alpha,\emptyset}(\emptyset)$ (if $\langle w(U_\gamma) \rangle \subseteq r$). Thus if $r \notin \{\emptyset, \langle w(U_\gamma) \rangle\}$, in both cases we would easily get $\rho_{U_\alpha,r}(r) > \rho_{U_\alpha,\emptyset}(\emptyset)$, and hence $\rho(U_\alpha, r) > \rho(U_\alpha, \emptyset)$. Since i witnesses $(U_\alpha, \emptyset) \cong (U_\alpha, i(\emptyset))$, so that $\rho(U_\alpha, \emptyset) = \rho(U_\alpha, i(\emptyset))$, setting $r := i(\emptyset)$ in the argument above we get that $i(\emptyset) \in \{\emptyset, \langle w(U_\gamma) \rangle\}$, as required. To see that both the possibilities can be realized, consider the identity map and the isomorphism $i: U_\alpha \rightarrow U_\alpha$ defined by

$$i(s) := \begin{cases} w(U_\gamma) \frown s & \text{if } s \in U_\gamma \\ t & \text{if } s = w(U_\gamma) \frown t. \end{cases}$$

The case of $\alpha = 0$ or α limit is similar: by construction, $\rho_{U_\alpha,i(\emptyset)}(\emptyset) = \rho_{U_\alpha,\emptyset}(\emptyset)$ independently of the value of $i(\emptyset)$, and therefore if $i(\emptyset) \neq \emptyset$ then $\rho(U_\alpha, i(\emptyset)) > \rho(U_\alpha, \emptyset)$, contradicting the fact that i witnesses $(U_\alpha, \emptyset) \cong (U_\alpha, i(\emptyset))$.

Finally we prove (f) by induction on α . First a technical claim.

CLAIM 10.5.1. *Let $\bar{i}: X \rightarrow U_\alpha$ be an arbitrary isomorphism.*

- (i) *Assume $\alpha = \gamma + 1$ is a successor ordinal, and set $\{\bar{\delta}_0, \bar{\delta}_1\} := \{\bar{i}^{-1}(\emptyset), \bar{i}^{-1}(\langle w(U_\gamma) \rangle)\}$, with $\bar{\delta}_0 < \bar{\delta}_1$. Then $i^{-1}(\{\emptyset, \langle w(U_\gamma) \rangle\}) = \{\bar{\delta}_0, \bar{\delta}_1\}$ for every isomorphism $i: X \rightarrow U_\alpha$.*
- (ii) *Assume that α is either 0 or a limit ordinal, and set $\bar{\delta} := \bar{i}^{-1}(\emptyset)$. Then $i(\bar{\delta}) = \emptyset$ for every isomorphism $i: X \rightarrow U_\alpha$.*

PROOF OF THE CLAIM. (i) Assume towards a contradiction that the claim fails for some isomorphism $i: X \rightarrow U_\alpha$. Using the fact that by (e) there is an automorphism of U_α sending $\langle w(U_\gamma) \rangle$ to \emptyset , we may assume without loss of generality that $i^{-1}(\emptyset) \neq \bar{\delta}_0, \bar{\delta}_1$. But then $i' := \bar{i} \circ i^{-1}$ would be an automorphism of U_α such that $i'(\emptyset) \notin \{\emptyset, \langle w(U_\gamma) \rangle\}$, contradicting part (e).

(ii) The case $\alpha = 0$ is trivial. The limit case can be treated similarly to the successor one: if $i(\bar{\delta}) \neq \emptyset$ for some isomorphism i , then we would also have an automorphism of U_α which does not fix its root \emptyset , contradicting again part (e). \square

We now come back to the proof of part (f) of the lemma. The case $\alpha = 0$ is trivial since U_0 is finite. Let $\alpha = \gamma + 1$. Fix an arbitrary isomorphism $\bar{i}: X \rightarrow U_\alpha$ and let $\bar{\delta}_0, \bar{\delta}_1$ be as in Claim 10.5.1(i). For the sake of definiteness, assume $\bar{\delta}_0 = \bar{i}^{-1}(\emptyset)$ — if this is not the case, just swap the following definitions of X_0 and X_1 . Set $X_0 := \bar{i}^{-1}(U_\gamma)$ and $X_1 := X \setminus X_0$, so that $\bar{\delta}_0 \in X_0$ and $\bar{\delta}_1 \in X_1$. Notice that by Claim 10.5.1(i) the definition of X_0, X_1 is independent of the choice of \bar{i} , therefore no choice is needed here. Since \bar{i} witnesses that both X_0 and X_1 are isomorphic to U_γ , by inductive hypothesis there are canonical isomorphisms $i_{X_0,\gamma}: X_0 \rightarrow U_\gamma$ and $i_{X_1,\gamma}: X_1 \rightarrow U_\gamma$. Notice that the isomorphism between X_j and U_γ induced by $\bar{i}|_{X_j}$ sends $\bar{\delta}_j$ to \emptyset , so by Claim 10.5.1 (applied to γ) and part (e) of the lemma we may assume without loss of generality that $i_{X_j,\gamma}(\bar{\delta}_j) = \emptyset$ (for $j = 0, 1$). Setting for $\delta \in X$

$$i_{X,\alpha}(\delta) := \begin{cases} i_{X_0,\gamma}(\delta) & \text{if } \delta \in X_0 \\ \langle w(U_\gamma) \rangle \frown i_{X_1,\gamma}(\delta) & \text{if } \delta \in X_1 \end{cases}$$

we get the desired canonical isomorphism between X and U_α .

Finally, assume that α is limit, and notice that in this case U_α is the disjoint union of U_0 and all subtrees of U_α with domain $\langle 3 + \beta \rangle \cap U_\beta = \{\langle 3 + \beta \rangle \cap t \mid t \in U_\beta\}$ for $\beta < \alpha$. Fix an arbitrary isomorphism $\bar{i}: X \rightarrow U_\alpha$, and let $\bar{\delta} := \bar{i}^{-1}(\emptyset)$ be as in Claim 10.5.1(ii). Set also, $X_\beta := \bar{i}^{-1}(\langle 3 + \beta \rangle \cap U_\beta)$ (for every $\beta < \alpha$) and $X_{-1} := X \setminus \bigcup_{\beta < \alpha} X_\beta$, so that

$$(10.3) \quad X_\beta \cong U_\beta \quad \text{and} \quad X_{-1} \cong U_0.$$

CLAIM 10.5.2. *For every isomorphism $i: X \rightarrow U_\alpha$ and every $\beta < \alpha$, $i(\bar{\delta}) = \emptyset$, $i(X_\beta) = \langle 3 + \beta \rangle \cap U_\beta$ and $i(X_{-1}) = U_0$.*

PROOF. Fix an arbitrary $\beta < \alpha$. Since $i(\bar{\delta}) = \emptyset$ by Claim 10.5.1(ii), we have that $i(X_\beta) = \{t \in U_\alpha \mid \langle \beta' \rangle \subseteq t\}$ for some $\beta' < 3 + \alpha = w(U_\alpha)$. Thus we just need to show that $\beta' = 3 + \beta$. Since X_β contains at least $|U_0| = 13$ points we have $\beta' \geq 3$, and thus $i(X_\beta) = \beta' \cap U_{\beta''}$ where $\beta' = 3 + \beta''$. Using i , \bar{i} , and (10.3), we get that $U_{\beta''} \cong X_\beta \cong U_\beta$ via an isomorphism sending \emptyset to itself, so $\beta = \beta''$ by part (c) of the lemma. Thus $\beta' = 3 + \beta$, as required. The final part concerning X_{-1} and U_0 follows from the preceding one, so we are done. \square

By Claims 10.5.1 and 10.5.2, the definition of $\bar{\delta}$, of the X_β 's, and of X_{-1} is independent of the chosen \bar{i} , so no choice is needed to define them. For every $\beta < \alpha$, let $\bar{\delta}_\beta$ be the unique point in X_β which is connected by an edge of X to $\bar{\delta}$, and apply the inductive hypothesis to get canonical isomorphisms $i_{X_{-1},0}: X_{-1} \rightarrow U_0$ and $i_{X_\beta,\beta}: X_\beta \rightarrow U_\beta$. Since \bar{i} induces an isomorphism between X_β and U_β sending $\bar{\delta}_\beta$ to \emptyset , by Claim 10.5.1 (applied to β) and part (e) of the lemma we may assume without loss of generality that $i_{X_\beta,\beta}(\bar{\delta}_\beta) = \emptyset$ as well. Similarly, using the fact that $\bar{i} \upharpoonright X_{-1}$ is an isomorphism between X_{-1} and U_0 sending $\bar{\delta}$ to \emptyset , we also have that $i_{X_{-1},0}(\bar{\delta}) = \emptyset$. Therefore setting

$$i_{X,\alpha}(\delta) := \begin{cases} i_{X_{-1},0}(\delta) & \text{if } \delta \in X_{-1} \\ \langle 3 + \beta \rangle \cap i_{X_\beta,\beta}(\delta) & \text{if } \delta \in X_\beta \text{ for some } \beta < \alpha, \end{cases}$$

we get the desired canonical isomorphism $i_{X,\alpha}: X \rightarrow U_\alpha$. \square

We next use some of the properties of the U_α 's described in Lemma 10.5 to obtain a rigidity property of \mathbb{G}_1 which will be crucial in the proof of the invariant universality of \mathbb{G}_1^κ .

LEMMA 10.6. *Every automorphism of \mathbb{G}_1 is the identity on G_0 .*

PROOF. Let j be an automorphism of \mathbb{G}_1 , and for every $s \in {}^{<\omega}\kappa$ let $U_s \subseteq G_1$ be the copy of $U_{\langle s \rangle}$ in \mathbb{G}_1 , that is the substructure of \mathbb{G}_1 with domain $\{(s, t) \in {}^{<\omega}\kappa \times {}^{<\omega}\kappa \mid t \in U_{\langle s \rangle}\}$ (see Definition 10.4). It suffices to prove that $j(s) = s$ for every $s \in {}^{<\omega}\kappa \subseteq G_0 \subseteq G_1$. First notice that $j({}^{<\omega}\kappa) = {}^{<\omega}\kappa$ because ${}^{<\omega}\kappa \subseteq G_1$ is the set of all elements of \mathbb{G}_1 of degree κ (by Lemma 10.5(a) and the fact that all vertices of the form s^- or \hat{s} have degree 2, see also the subsequent Lemma 10.8). In particular, j maps $G_1 \setminus {}^{<\omega}\kappa$ in itself. Consider the point $(s, \emptyset) \in G_1 \setminus {}^{<\omega}\kappa$, which is the root of U_s : since it has degree ≥ 4 and distance 2 from s , we must have $j(s, \emptyset) = (j(s), \emptyset) \in U_{j(s)}$ (recall that necessarily $j(G_1 \setminus {}^{<\omega}\kappa) \subseteq G_1 \setminus {}^{<\omega}\kappa$). It follows that $j \upharpoonright U_s$ is an isomorphism between U_s and $U_{j(s)}$ which sends the root of U_s to the root of $U_{j(s)}$. Therefore $\langle s \rangle = \langle j(s) \rangle$ by Lemma 10.5(c), and thus $j(s) = s$ by injectivity of $\langle \cdot \rangle$. \square

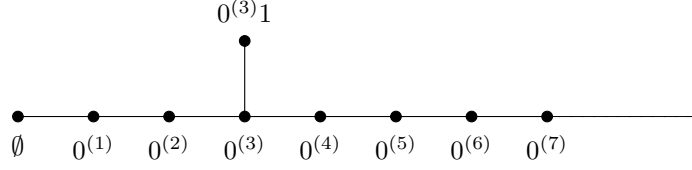
10.2. The combinatorial trees \mathbb{G}_S . As described at the beginning of this section, the combinatorial tree \mathbb{G}_S associated to some $S \in \text{Tr}(2 \times \kappa)$ is obtained by adding some forks to \mathbb{G}_1 .

Fix any injection $\theta: {}^{<\omega}2 \rightarrow \omega$ such that

$$(10.4a) \quad \theta(u) \text{ is odd for all } u \in {}^{<\omega}2,$$

$$(10.4b) \quad \theta(\emptyset) = 3 \text{ and } \theta(u) > \theta(\emptyset) \text{ for all } u \in {}^{<\omega}2 \setminus \{\emptyset\},$$

$$(10.4c) \quad |\theta(u) - \theta(v)| > 4 \cdot \max\{\text{lh } u, \text{lh } v\} \text{ for all distinct } u, v \in {}^{<\omega}2.$$

FIGURE 3. The fork F_u , when $\theta(u) = 3$ (i.e. $u = \emptyset$)

Such an injection can be easily constructed by induction on the length of $u \in {}^{<\omega}2$ by e.g. setting $\theta(\emptyset) := 3$ and by letting $\theta \upharpoonright {}^{n+1}2$ be an arbitrary injection into the set

$$\{\max\{\theta(u) \mid u \in {}^n 2\} + 5k(n+1) \mid 1 \leq k \leq 2^{n+1}\}.$$

For $u \in {}^{<\omega}2$ we define the **fork (coding u)** to be the graph F_u on

$$\{w \in {}^{<\omega}2 \mid w \subseteq 0^{(\omega)} \vee w = 0^{(\theta(u))} \smallfrown 1\}$$

connecting each sequence $w \neq \{\emptyset\}$ to its immediate predecessor $w^* = w \upharpoonright (\text{lh } w) - 1$ (Figure 3). Then F_u is a combinatorial tree consisting of three disjoint branches departing from $0^{(\theta(u))}$, which is the unique vertex of degree 3; one branch is infinite, one has length 1, and one has odd length $\theta(u) > 1$. Since any embedding respects degrees,

$$(10.5a) \quad u \neq v \Rightarrow \text{there is no embedding of } F_u \text{ into } F_v \text{ fixing } \emptyset,$$

$$(10.5b) \quad F_u \text{ is rigid, i.e. its unique automorphism is the identity,}$$

$$(10.5c) \quad u = v \Leftrightarrow F_u \cong F_v.$$

Using the fact that the domain of F_u is an infinite subset of ${}^{<\omega}2$, the graph F_u can be then identified with a graph on ω via a canonical bijection

$$(10.6) \quad e_u: \text{dom}(F_u) \rightarrow \omega,$$

sending \emptyset to 0.

For each $(u, s) \in {}^{<\omega}(2 \times \omega)$, we fix an isomorphic copy $F_{u,s}$ of F_u , so that the sets of vertices of $F_{u,s}$ and $F_{v,t}$ are disjoint whenever $(u, s) \neq (v, t)$, and such that each $F_{u,s}$ is also disjoint from the domain G_1 of \mathbb{G}_1 . More precisely, we let $F_{u,s} = \{(u, s)\} \times F_u$ be the graph on

$$\{(u, s, w) \mid w \subseteq 0^{(\omega)} \vee w = 0^{(\theta(u))} \smallfrown 1\}$$

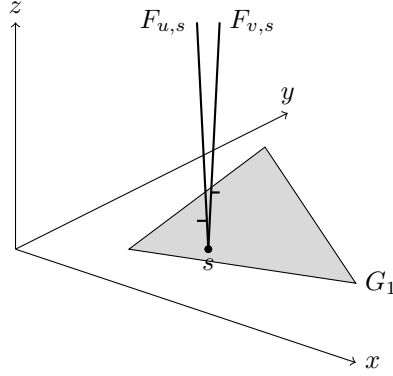
with set of edges defined by

$$(u, s, w) F_{u,s} (u, s, z) \Leftrightarrow w F_u z.$$

Following [LR05, Theorem 3.1] and [FMR11, Theorem 3.9], to each $S \in \text{Tr}(2 \times \kappa)$ we now associate the combinatorial tree \mathbb{G}_S obtained by joining \mathbb{G}_1 and the $F_{u,s}$ for $(u, s) \in S$ via the identification of any vertex $s \in {}^{<\omega}\kappa \subseteq G_1$ with each vertex of the form (u, s, \emptyset) . Thus the domain G_S of \mathbb{G}_S can be identified with

$$\begin{aligned} & {}^{<\omega}\kappa \uplus \{s^- \mid \emptyset \neq s \in {}^{<\omega}\kappa\} \uplus \{\hat{s} \mid \emptyset \neq s \in {}^{<\omega}\kappa\} \uplus \{(s, t) \in {}^{<\omega}\kappa \times {}^{<\omega}\kappa \mid t \in U_{\langle\langle s \rangle\rangle}\} \\ & \uplus \{(u, s, w) \mid (u, s) \in S \wedge (u, s, w) \in F_{u,s} \wedge w \neq \emptyset\}. \end{aligned}$$

(Figure 4 is a clumsy attempt to visualize \mathbb{G}_S in the three-dimensional space: \mathbb{G}_1 is the grey area in the xy -plane, $s \in {}^{<\omega}\kappa$ is a vertex of \mathbb{G}_1 in the set ${}^{<\omega}\kappa \subseteq G_0 \subseteq G_1$, and the forks $F_{u,s}, F_{v,s}$ are growing vertically.)

FIGURE 4. The graph \mathbb{G}_S .

To simplify the presentation, we also introduce the following notation for the relevant subsets of the domain G_S of \mathbb{G}_S (when necessary, each of these sets is also identified with the corresponding substructure of \mathbb{G}_S):

$$(10.7a) \quad \text{Seq}(\mathbb{G}_S) := \{s \in G_S \mid s \in {}^{<\omega}\kappa\};$$

$$(10.7b) \quad \text{Seq}^-(\mathbb{G}_S) := \{s^- \in G_S \mid s \in {}^{<\omega}\kappa\};$$

$$(10.7c) \quad \widehat{\text{Seq}}(\mathbb{G}_S) := \{\hat{s} \in G_S \mid s \in {}^{<\omega}\kappa\};$$

$$(10.7d) \quad U_s(\mathbb{G}_S) := \{(s, t) \in {}^{<\omega}\kappa \times {}^{<\omega}\kappa \mid t \in U_{\langle\langle s \rangle\rangle}\} \text{ for every } s \in {}^{<\omega}\kappa;$$

$$(10.7e) \quad U(\mathbb{G}_S) := \bigcup \{U_s \mid s \in {}^{<\omega}\kappa\};$$

$$(10.7f) \quad F'_{u,s}(\mathbb{G}_S) := \{(u, s, w) \in F_{u,s} \mid w \neq \emptyset\} \text{ for every } (u, s) \in S;$$

$$(10.7g) \quad F(\mathbb{G}_S) := \bigcup \{F'_{u,s} \mid (u, s) \in S\}.$$

REMARK 10.7. Notice that both the subsets and the corresponding substructures of \mathbb{G}_S defined in (10.7a)–(10.7f) do not depend at all on the specific $S \in \text{Tr}(2 \times \kappa)$ under consideration, but only on the parameters s and u (when they appear in the notation). Therefore, to further simplify the notation we can safely set

$$\text{Seq} := \text{Seq}(\mathbb{G}_S) \quad (\text{for some/any } S \in \text{Tr}(2 \times \kappa)),$$

and define in a similar way the structures Seq^- , $\widehat{\text{Seq}}$, U_s , U , and $F'_{u,s}$.

In particular, we have $G_0 = \text{Seq} \uplus \text{Seq}^-$, $G_1 = G_0 \uplus \widehat{\text{Seq}} \uplus U$, and $G_S = G_1 \uplus F$. Finally, the graph \mathbb{G}_S is then further identified in a canonical way with its copy on κ (which is thus an element of CT_κ), using the bijections $\langle\langle \cdot \rangle\rangle$, $\langle \cdot, \cdot \rangle$, and e_u in the obvious way.

We end this section by listing some properties of the vertices of \mathbb{G}_S in term of distances and degrees which will be useful in the subsequent proofs (Sections 11 and 12), leaving to the reader to check their validity (using when necessary Lemma 10.5). Given a combinatorial tree $G \in \text{CT}_\kappa$, a **\mathbb{Z} -chain of G** is a sequence $\langle x_z \mid z \in \mathbb{Z} \rangle$ of pairwise distinct elements of G such that $x_z \mathbf{E}^G x_{z+1}$ for every $z \in \mathbb{Z}$.

LEMMA 10.8. *Let $S \in \text{Tr}(2 \times \kappa)$.*

- (a) *The vertices in Seq are the unique vertices of \mathbb{G}_S having degree κ ; they are also the unique vertices of \mathbb{G}_S with degree ≥ 4 all of whose neighbors have degree $= 2$. The distance between two elements of Seq is always even.*

- (b) *The vertices in Seq^- are the unique vertices of \mathbb{G}_S with degree 2 and all of whose neighbors are in Seq .*
- (c) *The vertices of the form $(u, s, 0^{(\theta(u))})$ (for some $(u, s) \in S$) are the unique vertices of \mathbb{G}_S having degree (at least) 3, odd distance from the vertices in Seq , and belonging to a \mathbb{Z} -chain of \mathbb{G}_S . All other vertices in $F'_{u,s}$ have degree at most 2.*
- (d) *The vertices in F are the unique vertices in \mathbb{G}_S with the following three properties:*
 - *they have degree < 4 ;*
 - *all their neighbors are either in Seq or have degree < 4 as well;*
 - *at least one of their neighbors has degree < 4 .*
- (e) *The vertices in U are the unique vertices of \mathbb{G}_S with the following two properties:*
 - *they have degree $\neq 2$;*
 - *they are adjacent to a vertex of degree ≥ 4 which in turn has another neighbor (distinct from the described one) with degree ≥ 4 as well.**They do not belong to any \mathbb{Z} -chain of \mathbb{G}_S , and they have degree strictly smaller than κ .*
- (f) *The vertices in $\widehat{\text{Seq}}$ are the unique vertices in \mathbb{G}_S with the following three properties:*
 - *they have degree 2;*
 - *at least one of their neighbors has degree ≥ 4 and belongs to U ;*
 - *all their neighbors have degree ≥ 4 .*

REMARK 10.9. In many cases, the list of conditions in Lemma 10.8 for characterizing the vertices belonging to the various substructures of \mathbb{G}_S is overkill. For example, by suppressing the condition “all their neighbors have degree ≥ 4 ” in (f) we would still get a correct characterization of the vertices in $\widehat{\text{Seq}}$; however, such extra requirement is what makes the conjunction of the conditions from (f) incompatible with the conjunction of the conditions from (d). This “pairwise incompatibility” of the characterizations of the various substructures of \mathbb{G}_S will become a very useful feature when we will have to render them with corresponding $\mathcal{L}_{\kappa+\kappa}$ -formulæ in order to prove our invariant universality result — see Remark 12.4.

11. Completeness

In this section we will that from any tree T on $2 \times 2 \times \kappa$ witnessing that $R = p[T]$ is a κ -Souslin quasi-order, one can build a function $f_T: {}^\omega 2 \rightarrow \text{CT}_\kappa$ that reduces R to $\sqsubseteq_{\text{CT}}^\kappa$. The function f_T is constructed in three steps:

- firstly the tree T is replaced by a better tree \tilde{T} called *faithful representation of R* (Lemma 11.4);
- a function $\Sigma_T: {}^\omega 2 \rightarrow \text{Tr}(2 \times \kappa)$ is constructed, so that Σ_T reduces R to \leq_{max}^κ , where the latter is a quasi-order on $\text{Tr}(2 \times \kappa)$ which is independent of T (Lemma 11.6);
- finally, we compose Σ_T with the map $\text{Tr}(2 \times \kappa) \rightarrow \text{CT}_\kappa$, $S \mapsto \mathbb{G}_S$ defined in Section 10, and then check that the resulting function $f_T(x) = \mathbb{G}_{\Sigma_T(x)}$ is the required reduction (Theorem 11.8).

11.1. Faithful representations of κ -Souslin quasi-orders. Let $\langle \cdot, \cdot \rangle: \text{Ord} \times \text{Ord} \rightarrow \text{Ord}$ be the pairing function as in (2.1), and let

$$\varrho(\alpha, \beta) := 2 + \langle \alpha, \beta \rangle.$$

Then $\varrho: \text{Ord} \times \text{Ord} \rightarrow \text{Ord} \setminus \{0, 1\}$ is a bijection that maps $\kappa \times \kappa$ onto $\kappa \setminus \{0, 1\}$ for all cardinals $\kappa \geq \omega$, and it satisfies $\varrho(n, m) > n, m$ for all $n, m \in \omega$. Thus the map

$$(11.1) \quad \bar{\varrho}: {}^{\leq \omega}(\text{Ord} \times \text{Ord}) \rightarrow {}^{\leq \omega}(\text{Ord} \setminus \{0, 1\})$$

defined using ϱ coordinate-wise

$$\bar{\varrho}(\langle (\alpha_0, \beta_0), (\alpha_1, \beta_1), \dots \rangle) := \langle \varrho(\alpha_0, \beta_0), \varrho(\alpha_1, \beta_1), \dots \rangle$$

is a bijection.

DEFINITION 11.1. For κ an infinite cardinal, let

$$\mathbb{T}_\kappa := \text{Tr}(2 \times 2 \times \kappa)$$

and $\mathbb{T} := \bigcup_{\kappa \in \text{Card}} \mathbb{T}_\kappa$.

Given a κ -Souslin quasi-order R on ${}^\omega 2$, we would like to have a witness $\tilde{T} \in \mathbb{T}_\kappa$ of the fact that R is κ -Souslin which also witnesses transitivity and reflexivity of R at all finite levels in a uniform way. To be more specific, we want reflexivity to be witnessed by almost all elements in ${}^\omega \kappa$ and that witnesses of transitivity are given by $\bar{\varrho}$.

DEFINITION 11.2. Let κ be an infinite cardinal, and R be a κ -Souslin quasi-order on ${}^\omega 2$. A tree $\tilde{T} \in \mathbb{T}_\kappa$ is called **faithful representation** of R if the following conditions hold:

- (1) $R = p[\tilde{T}]$;
- (2) $\forall u, n, s \left(\text{lh } u = \text{lh } s + 1 \Rightarrow (u, u, n \frown s) \in \tilde{T} \right)$ (reflexivity);
- (3) $\forall u, v, w, s, t \left((u, v, s), (v, w, t) \in \tilde{T} \Rightarrow (u, w, \bar{\varrho}(s, t)) \in \tilde{T} \right)$ (transitivity);
- (4) $\forall u, v \left((u, v, 0^{(\text{lh } u)}) \in \tilde{T} \Rightarrow u = v \right)$.

REMARK 11.3. Notice that the statement “ $\tilde{T} \in \mathbb{T}_\kappa$ is a faithful representation (of $R = p[\tilde{T}]$)” is absolute for transitive models M of **ZF** containing \tilde{T} and such that $\kappa \in \text{Card}^M$. Moreover, a faithful representation \tilde{T} not only “combinatorially” reflects the properties of the quasi-order $R = p[\tilde{T}]$, but it makes the κ -Souslinness of R more robust because in every **ZF**-model M as above the tree \tilde{T} continues to define a canonical κ -Souslin quasi-order $R_{\tilde{T}}^M := (p[\tilde{T}])^M$ which is coherent with R in the following sense:

$$(11.2) \quad \forall x, y \in {}^\omega 2 \cap M \ (x R y \Leftrightarrow x R_{\tilde{T}}^M y).$$

In fact, by Definition 11.2(2)–(3) we get that for all $x, y, z \in ({}^\omega 2)^M$:

- (i) every $\xi \in ({}^\omega \kappa)^M$ with $\xi(0) \in \omega$ witnesses $(x, x) \in (p[\tilde{T}])^M$;
- (ii) if $\xi_0, \xi_1 \in ({}^\omega \kappa)^M$ witness, respectively, $(x, y) \in (p[\tilde{T}])^M$ and $(y, z) \in (p[\tilde{T}])^M$, then $\bar{\varrho}(\xi_0, \xi_1)$ witnesses $(x, z) \in (p[\tilde{T}])^M$.

Therefore $R_{\tilde{T}}^M = (p[\tilde{T}])^M$ is a quasi-order. The coherence condition (11.2) easily follows by absoluteness of the existence of infinite branches through \tilde{T} .

These absoluteness properties of faithful representations \tilde{T} will be exploited in Section 14.2.

We are now going to show that every κ -Souslin quasi-order R on ${}^\omega 2$ admits a faithful representation \tilde{T} . Indeed, the following variation of the construction given in the proof of [LR05, Theorem 2.4] shows how to construct such a \tilde{T} starting from an arbitrary $T \in \mathbb{T}_\kappa$ with $R = p[T]$.

For $T \in \mathbb{T}_\kappa$ let

$$\hat{T} := T \cup \{ (u, u, s) \in {}^{<\omega} 2 \times {}^{<\omega} 2 \times {}^{<\omega} \kappa \mid \text{lh } u = \text{lh } s \}.$$

Then $\hat{T} \in \mathbb{T}_\kappa$, and $p[T] = p[\hat{T}]$ whenever $p[T]$ is a reflexive relation on ${}^\omega 2$. Recall from (2.4) that if $u \neq \emptyset$ then $u^* = u \upharpoonright (\text{lh } u - 1)$. Inductively define:

$$\begin{aligned} \tilde{T}_0 &:= \{ (\emptyset, \emptyset, \emptyset) \} \cup \{ (u, v, 0 \frown s) \mid (u^*, v^*, s) \in \hat{T} \} \\ \tilde{T}_{n+1} &:= \{ (\emptyset, \emptyset, \emptyset) \} \cup \{ (u, v, (n+1) \frown s) \mid (u, v, n \frown s) \in \tilde{T}_n \} \cup \\ &\quad \cup \{ (u, w, (n+1) \frown \bar{\varrho}(s, t)) \mid \exists v \left((u, v, n \frown s), (v, w, n \frown t) \in \tilde{T}_n \right) \}. \end{aligned}$$

It is immediate to check that for all $n \in \omega$

- $s \neq \emptyset \wedge (u, v, s) \in \tilde{T}_n \Rightarrow s(0) = n$,
- $\tilde{T}_n \in \mathbb{T}_\kappa$,

- $s \neq \emptyset \wedge (u, v, s) \in \hat{T} \Rightarrow (u, v, n \wedge s^*) \in \tilde{T}_n$, and in particular $p[\hat{T}] \subseteq p[\tilde{T}_n]$.

Then

$$(11.3) \quad \tilde{T} := \left(\bigcup_{n \in \omega} \tilde{T}_n \right) \setminus \{(u, v, 0^{(k)}) \mid u, v \in {}^k 2 \wedge u \neq v \wedge k > 0\} \in \mathbb{T}_\kappa.$$

By construction, if $(u, v, s) \in \tilde{T}$ and $s \neq \emptyset$ then $s(0) \in \omega$ and $(u, v, s) \in \tilde{T}_{s(0)} \setminus \bigcup_{j < s(0)} \tilde{T}_j$, so

$$\tilde{T} = \left(\bigcup_{n \geq 1} \tilde{T}_n \right) \cup (\tilde{T}_0 \setminus \{(u, v, 0^{(k)}) \mid u, v \in {}^k 2 \wedge u \neq v \wedge k > 0\}).$$

LEMMA 11.4. *Let κ be an infinite cardinal, and let $T \in \mathbb{T}_\kappa$. If $R = p[T]$ is a quasi-order, then the tree $\tilde{T} \in \mathbb{T}_\kappa$ defined in (11.3) is a faithful representation of R .*

PROOF. We have to show that \tilde{T} satisfies the four conditions (1)–(4) of Definition 11.2.

The tree \tilde{T} satisfies (4), and it satisfies also (2) by definition of \tilde{T} . To prove (3), we can assume that $s, t \neq \emptyset$ and set $n := s(0)$ and $m := t(0)$, so that $(u, v, s) \in \tilde{T}_n$ and $(v, w, t) \in \tilde{T}_m$. We also let $\bar{s} := \langle s(i) \mid 1 \leq i < \text{lh } s \rangle$ and $\bar{t} := \langle t(i) \mid 1 \leq i < \text{lh } t \rangle$. Then $(u, v, k \wedge \bar{s}), (v, w, k \wedge \bar{t}) \in \tilde{T}_k$ for all $k \geq \max\{n, m\}$, and in particular for $k := \varrho(n, m) - 1$. Therefore

$$(u, w, (k+1) \wedge \bar{\varrho}(\bar{s}, \bar{t})) = (u, w, \varrho(n, m) \wedge \bar{\varrho}(\bar{s}, \bar{t})) = (u, w, \bar{\varrho}(s, t)) \in \tilde{T}_{\varrho(n, m)} \subseteq \bigcup_{n \in \omega} \tilde{T}_n.$$

Since ϱ never takes value 0 then $\bar{\varrho}(s, t) \neq 0^{(\text{lh } u)}$, thus $(u, w, \bar{\varrho}(s, t)) \in \tilde{T}$, as required.

It remains to prove (1). One direction is easy: $R = p[\hat{T}]$ (since R is reflexive) and $p[\hat{T}] = p[\tilde{T}_0] \subseteq p[\tilde{T}_1] \subseteq p[\tilde{T}]$. Since $p[\tilde{T}] \subseteq p[\bigcup_n \tilde{T}_n]$ it is enough to prove that $p[\bigcup_n \tilde{T}_n] \subseteq R$, so we may assume that $(x, y) \in p[\bigcup_n \tilde{T}_n]$ and let $\xi \in {}^\omega \kappa$ be a witness of this. Then $\forall k (x \upharpoonright k, y \upharpoonright k, \xi \upharpoonright k) \in \tilde{T}_{\xi(0)}$, so that $(x, y) \in p[\tilde{T}_{\xi(0)}]$. Therefore it is enough to prove by induction on n that

CLAIM 11.4.1. *For every $n \in \omega$, $p[\tilde{T}_n] \subseteq R$, and in fact $p[\tilde{T}_n] = R$ since $R \subseteq p[\tilde{T}_0] \subseteq p[\tilde{T}_n]$.*

PROOF OF CLAIM. The proof is by induction on $n \in \omega$. The case $n = 0$ is obvious as $p[\tilde{T}_0] = p[\hat{T}] = R$ (by reflexivity of R), so assume $p[\tilde{T}_n] \subseteq R$, choose an arbitrary $(x, y) \in p[\tilde{T}_{n+1}]$ and let $\bar{\xi} \in {}^\omega \kappa$ be such that $(x, y, (n+1) \wedge \bar{\xi}) \in [\tilde{T}_{n+1}]$. Because of the definition of \tilde{T}_{n+1} we have to distinguish two cases:

Case 1: $\exists^\infty k [(x \upharpoonright k, y \upharpoonright k, n \wedge \bar{\xi} \upharpoonright (k-1))] \in \tilde{T}_n$. Then $(x, y, n \wedge \bar{\xi}) \in [\tilde{T}_n]$, so that $(x, y) \in p[\tilde{T}_n] \subseteq R$ (by inductive hypothesis).

Case 2: $\forall^\infty k \exists v_k [(x \upharpoonright k, v_k, n \wedge \bar{\xi}_0 \upharpoonright (k-1)), (v_k, y \upharpoonright k, n \wedge \bar{\xi}_1 \upharpoonright (k-1))] \in \tilde{T}_n$, where $\bar{\xi}_0, \bar{\xi}_1 \in {}^\omega \kappa$ are the unique elements such that $\bar{\xi} = \bar{\varrho}(\bar{\xi}_0, \bar{\xi}_1)$. Now notice that the collection of all possible v_k 's as above form an infinite finitely-branching tree (there are infinitely many v_k 's because such witnesses must be distinct for different $k > 0$ as $\text{lh } v_k = k$), so by König's lemma there is an infinite branch $z \in {}^\omega 2$ through it which has the property that $(x \upharpoonright k, z \upharpoonright k, n \wedge \bar{\xi}_0 \upharpoonright (k-1)), (z \upharpoonright k, y \upharpoonright k, n \wedge \bar{\xi}_1 \upharpoonright (k-1)) \in \tilde{T}_n$ for every $k > 0$. Therefore $\xi_0 := n \wedge \bar{\xi}_0$ and $\xi_1 := n \wedge \bar{\xi}_1$ witness $(x, z), (z, y) \in p[\tilde{T}_n] \subseteq R$, so that $(x, y) \in R$ by the transitivity of R . \square

This concludes the proof of the lemma. \square

11.2. The quasi-order \leq_{\max} and the reduction Σ_T .

DEFINITION 11.5. Given trees $S, S' \in \text{Tr}(2 \times \kappa)$, let

$$S \leq_{\max}^\kappa S'$$

if and only if there is a Lipschitz (i.e. a monotone and length-preserving) function $\varphi: {}^{<\omega} \kappa \rightarrow {}^{<\omega} \kappa$ such that for all $u \in {}^{<\omega} 2$ and $s \in {}^{<\omega} \kappa$ of the same length

$$(u, s) \in S \Rightarrow (u, \varphi(s)) \in S'.$$

If φ can be taken to be injective, we write $S \preceq_{\max}^{\kappa} S'$.

As observed in [LR05], if we restrict our attention to *normal* trees on $2 \times \kappa$ (that is trees S such that $(u, t) \in S$ whenever there is $s \in {}^{\text{lh } t}\kappa$ such that t is pointwise bigger than s and $(u, s) \in S$) then the quasi-orders \leq_{\max}^{κ} and \preceq_{\max}^{κ} coincide. However, unlike the case $\kappa = \omega$ considered in [LR05], in the uncountable case we cannot require the tree \tilde{T} defined in (11.3) to be normal, so the two quasi-orders \leq_{\max}^{κ} and \preceq_{\max}^{κ} must be dealt with separately.

DEFINITION 11.6. For $T \in \mathbb{T}_{\kappa}$, let $\Sigma_T: {}^{\omega}2 \rightarrow \text{Tr}(2 \times \kappa)$ be defined as

$$\Sigma_T(x) := \left\{ (u, s) \mid (u, x \upharpoonright \text{lh } u, s) \in \tilde{T} \right\},$$

where \tilde{T} is as in (11.3).

Recall that $\langle \cdot \rangle: {}^{<\omega}\text{Ord} \rightarrow \text{Ord}$ is the bijection of (2.2) and that it maps ${}^{<\omega}\kappa$ onto κ .

LEMMA 11.7. Let $T \in \mathbb{T}_{\kappa}$ be such that $R = p[T]$ is a quasi-order.

- (a) Σ_T simultaneously reduces R to \leq_{\max}^{κ} and \preceq_{\max}^{κ} . In particular, both \leq_{\max}^{κ} and \preceq_{\max}^{κ} are complete for κ -Souslin quasi-orders.
- (b) If $x, y \in {}^{\omega}2$ are such that $x R y$, then there is a witness φ of $\Sigma_T(x) \preceq_{\max}^{\kappa} \Sigma_T(y)$ such that $\langle s \rangle \leq \langle \varphi(s) \rangle$ for every $s \in {}^{<\omega}\kappa$.
- (c) Σ_T is injective.

PROOF. (a) Since \preceq_{\max}^{κ} refines \leq_{\max}^{κ} , it is enough to show that if $R = p[T]$ is a quasi-order (so that \tilde{T} is a faithful representation of R by Lemma 11.4) then for every $x, y \in {}^{\omega}2$,

$$\Sigma_T(x) \leq_{\max}^{\kappa} \Sigma_T(y) \Rightarrow x R y \Rightarrow \Sigma_T(x) \preceq_{\max}^{\kappa} \Sigma_T(y).$$

The proof is identical to the one of [LR05, Theorem 2.5]. Suppose first that φ witnesses $\Sigma_T(x) \leq_{\max}^{\kappa} \Sigma_T(y)$. Let $\xi := \bigcup_{k \in \omega} \varphi(0^{(k)})$. By reflexivity of \tilde{T} (Definition 11.2(2)), $(x \upharpoonright k, 0^{(k)}) \in \Sigma_T(x)$, and hence $(x \upharpoonright k, \varphi(0^{(k)})) \in \Sigma_T(y)$ for all k . But this means that $(x, y, \xi) \in [\tilde{T}]$, and hence $(x, y) \in R = p[\tilde{T}]$ (where for the last equality we use Definition 11.2(1)).

Assume now that $\xi \in {}^{\omega}\kappa$ witnesses $(x, y) \in p[\tilde{T}] = R$. Then $\xi(0) \in \omega$. For $s \in {}^{<\omega}\kappa$, let

$$(11.4) \quad \varphi(s) := \bar{\varrho}(s, \xi \upharpoonright \text{lh } s).$$

Since the function $\bar{\varrho}$ used to define $\bar{\varrho}$ is injective, then φ is injective as well. Suppose now that $(u, s) \in \Sigma_T(x)$ and let $k := \text{lh } u = \text{lh } s$. Then $s(0) \in \omega$ and $(u, x \upharpoonright k, s) \in \tilde{T}$. On the other hand $(x \upharpoonright k, y \upharpoonright k, \xi \upharpoonright k) \in \tilde{T}$, therefore $(u, y \upharpoonright k, \bar{\varrho}(s, \xi \upharpoonright k)) \in \tilde{T}$ by transitivity of \tilde{T} (Definition 11.2(3)), so $(u, \varphi(s)) \in \Sigma_T(y)$.

(b) The map φ defined in (11.4) will do.

(c) This follows from the fact that \tilde{T} satisfies Definition 11.2(4), as if $x \neq y$ and $k \in \omega$ is such that $x \upharpoonright k \neq y \upharpoonright k$, then $(x \upharpoonright k, 0^{(k)}) \in \Sigma_T(x) \setminus \Sigma_T(y)$. \square

11.3. Reducing \leq_{\max} to $\sqsubset_{\text{CT}}^{\kappa}$. Now we are ready to prove the main result of this section. Given an infinite cardinal κ and $T \in \mathbb{T}_{\kappa}$, define the function

$$(11.5) \quad f_T: {}^{\omega}2 \rightarrow \text{CT}_{\kappa}, \quad x \mapsto \mathbb{G}_{\Sigma_T(x)},$$

where Σ_T is as in Definition 11.6 and the combinatorial tree $\mathbb{G}_{\Sigma_T(x)}$ associated to $\Sigma_T(x) \in \text{Tr}(2 \times \kappa)$ is defined as in Section 10.2.

THEOREM 11.8. Let κ be an infinite cardinal and $T \in \mathbb{T}_{\kappa}$. If $R = p[T]$ is a quasi-order, then the map f_T defined in (11.5) is such that:

- (a) f_T reduces R to the embeddability relation $\sqsubset_{\text{CT}}^{\kappa}$;
- (b) f_T reduces $=$ on ${}^{\omega}2$ to the isomorphism relation $\cong_{\text{CT}}^{\kappa}$.

In particular, $\sqsubseteq_{\text{CT}}^\kappa$ is complete for κ -Souslin quasi-orders on ${}^\omega 2$, i.e. every κ -Souslin quasi-order on ${}^\omega 2$ is reducible to the embeddability relation on CT_κ .

PROOF. The proof of (a) is similar to the ones of [LR05, Theorem 3.1] and [FMR11, Theorem 3.9], but it is simplified a little bit by our different choice of the map θ . Let $x, y \in {}^\omega 2$, and assume first that $x R y$. By Lemma 11.7(b), there is a $\varphi: {}^{<\omega}\kappa \rightarrow {}^{<\omega}\kappa$ witnessing $\Sigma_T(x) \preceq_{\max}^\kappa \Sigma_T(y)$ such that $\langle\langle s \rangle\rangle \leq \langle\langle \varphi(s) \rangle\rangle$ for every $s \in {}^{<\omega}\kappa$. Set

- $i(s) := \varphi(s)$, $i(s^-) := (\varphi(s))^-$, and $i(\hat{s}) := \widehat{\varphi(s)}$ for every $s \in {}^{<\omega}\kappa$;
- $i(s, t) := (\varphi(s), t)$ for every $s \in {}^{<\omega}\kappa$ and $t \in U_{\langle\langle s \rangle\rangle}$ (this definition is well given by $\langle\langle s \rangle\rangle \leq \langle\langle \varphi(s) \rangle\rangle$ and Proposition 10.5(b));
- $i(u, s, w) := (u, \varphi(s), w)$ for every $(u, s) \in \Sigma_T(x)$, $(u, s, w) \in F_{u,s}$, and $w \neq \emptyset$ (this definition is well given because $(u, s) \in \Sigma_T(x) \Rightarrow (u, \varphi(s)) \in \Sigma_T(y)$ by our choice of φ).

It is easy to check that i is the desired embedding of $f_T(x)$ into $f_T(y)$.

Conversely, let j be an embedding of $f_T(x) = \mathbb{G}_{\Sigma_T(x)}$ into $f_T(y) = \mathbb{G}_{\Sigma_T(y)}$. By Lemma 11.7 it is enough to show that $\Sigma_T(x) \preceq_{\max}^\kappa \Sigma_T(y)$. Since embeddings cannot decrease degrees, by Lemma 10.8(a) we get $j(\text{Seq}) \subseteq \text{Seq}$. Moreover, Lemma 10.8 implies that each vertex of the form $(u, s, 0^{(\theta(u))})$ is sent into a vertex of the same form because the properties characterizing these vertices listed in Lemma 10.8(c) are preserved by embeddings (and $j(\text{Seq}) \subseteq \text{Seq}$). In particular it follows that $j(G_0 \cup F(\mathbb{G}_{\Sigma_T(x)})) \subseteq G_0 \cup F(\mathbb{G}_{\Sigma_T(y)})$. Since $(\emptyset, \emptyset, 0^{(3)})$ has distance 3 (which is the minimal value attained by θ) from $\emptyset \in \text{Seq}$ by (10.4b), we get $j(\emptyset, \emptyset, 0^{(3)}) = (\emptyset, \emptyset, 0^{(3)})$, whence $j(\emptyset) = \emptyset$. Arguing by induction on $\text{lh } s$ (and using injectivity of j and $j(\text{Seq}) \subseteq \text{Seq}$) we then get that $\varphi := j \upharpoonright \text{Seq}: {}^{<\omega}\kappa \rightarrow {}^{<\omega}\kappa$ is an injective Lipschitz map.

CLAIM 11.8.1. For each $(u, s) \in \Sigma_T(x)$, $j(u, s, 0^{(\theta(u))}) = (u, \varphi(s), 0^{(\theta(u))})$ (recall that $\varphi(s) := j(s)$).

PROOF OF CLAIM. Recall that j must send $(u, s, 0^{(\theta(u))})$ into a vertex of the same form, so let $(v, t) \in \Sigma_T(y)$ be such that $j(u, s, 0^{(\theta(u))}) = (v, t, 0^{(\theta(v))})$. Note that $\theta(u)$ is the distance in $f_T(x)$ between s and $(u, s, 0^{(\theta(u))})$ and $\theta(v)$ is the distance in $f_T(y)$ between t and $(v, t, 0^{(\theta(v))})$. Moreover, the path in $f_T(x)$ between the nodes s and $(u, s, 0^{(\theta(u))})$ is mapped by j to the (unique) path in $f_T(y)$ between the vertices $\varphi(s) := j(s)$ and $(v, t, 0^{(\theta(v))})$, which necessarily passes through t . This implies that $\theta(u) - \theta(v)$ is the distance in $f_T(y)$ between $\varphi(s)$ and t : but such distance is $\leq 4 \cdot \max\{\text{lh } \varphi(s), \text{lh } t\}$ (because the latter is an upper bound for the length of the path in $f_T(y)$ which goes from $\varphi(s)$ to \emptyset and then back to t), and since $\text{lh } u = \text{lh } s = \text{lh } \varphi(s)$ and $\text{lh } v = \text{lh } t$, this implies $u = v$ by (10.4c), whence also $\varphi(s) = t$ because such vertices have then distance $\theta(u) - \theta(v) = 0$. \square

Claim 11.8.1 easily implies that $(u, s) \in \Sigma_T(x) \Rightarrow (u, \varphi(s)) \in \Sigma_T(y)$, i.e. that φ witnesses $\Sigma_T(x) \preceq_{\max}^\kappa \Sigma_T(y)$.

(b) Fix an isomorphism j between $f_T(x) = \mathbb{G}_{\Sigma_T(x)}$ and $f_T(y) = \mathbb{G}_{\Sigma_T(y)}$. It follows from parts (a), (c), and (e) of Lemma 10.8 that $j(\text{Seq}) = \text{Seq}$, $j(G_0) = G_0$, and $j(G_1) = G_1$. In particular, $j \upharpoonright G_1$ is an automorphism of \mathbb{G}_1 . Therefore by Lemma 10.6, $j \upharpoonright G_0$ is the identity, and so is $\varphi := j \upharpoonright \text{Seq}$. Thus the second part of the proof of (a) shows that $\Sigma_T(x) \subseteq \Sigma_T(y)$ (because φ is now the identity map). Replacing j with j^{-1} in this argument, we obtain $\Sigma_T(y) \subseteq \Sigma_T(x)$, so that $\Sigma_T(x) = \Sigma_T(y)$. Since $x \mapsto \Sigma_T(x)$ is injective by Lemma 11.7(c), this implies that f_T reduces $=$ to \cong , as required. \square

REMARK 11.9. Notice that the second half of the proof of part (a) shows that $x R y$ whenever there is an embedding of the subgraph of $f_T(x)$ with domain $G_0 \cup F(\mathbb{G}_{\Sigma_T(x)})$ into the subgraph of $f_T(y)$ with domain $G_0 \cup F(\mathbb{G}_{\Sigma_T(y)})$. This feature will be used in Sections 13 and 16.2.1 — see the proofs of Theorems 13.3 and 16.8.

11.4. Some absoluteness results. By closely inspecting the constructions provided in Sections 10 and 11, one easily sees that the definition of the map f_T from (11.5) only requires the knowledge of the parameters T and κ , and that such definition is uniform in those parameters and independent of the transitive model of ZF we are working in. In fact, since the tree \tilde{T} in (11.3) is definable from T and κ , then the function sending (κ, T) to the map f_T is definable (without parameters) via an LST-formula, which moreover is absolute for transitive models of ZF. To be more precise, let M be an arbitrary transitive model of ZF, κ be a cardinal in M , and $T \in (\mathbb{T}_\kappa)^M$: then, working inside M , we can define the function $f_T^M := f_T$ as in (11.5), which continues to be a reduction of $R^M := (p[T])^M$ to $(\preceq_{CT}^\kappa)^M$ as long as R^M is a quasi-order in M (because Theorem 11.8, which is proved in ZF, holds in M). With this notation, we then get the definability and absoluteness results briefly discussed below, which will be used in Section 14.2.

PROPOSITION 11.10. *There is an LST-formula $\Psi_{f_T}(x_0, x_1, z_0, z_1)$ with the following properties.*

- (a) *For every transitive model M of ZF, $\kappa \in \text{Card}^M$ and $T \in (\mathbb{T}_\kappa)^M$, the formula $\Psi_{f_T}(x_0, x_1, \kappa, T)$ defines in M (the graph of) f_T^M , that is: for every $x \in (\omega 2)^M$ and $X \in (CT_\kappa)^M$*

$$f_T^M(x) = X \Leftrightarrow M \models \Psi_{f_T}[x, X, \kappa, T].$$

- (b) *Let N be another transitive model of ZF with $\kappa \in \text{Card}^N$ and $T \in (\mathbb{T}_\kappa)^N$, and let f_T^N be the function defined in N by the formula $\Psi_{f_T}(x_0, x_1, \kappa, T)$ — see part (a). Then*

$$\forall x \in (\omega 2)^M \cap (\omega 2)^N (f_T^M(x) = f_T^N(x)),$$

i.e. f_T^M and f_T^N coincide on the common part of their domain.

SKETCH OF THE PROOF. Indeed, $\Psi_{f_T}(x_0, x_1, z_0, z_1)$ is the formalization in the language of set theory of the construction of the combinatorial tree $x_1 := f_T(x_0) = \mathbb{G}_{\Sigma_T(x_0)} \in CT_\kappa$ starting from the parameters $x_0 \in \omega 2$, $z_0 := \kappa$, and $z_1 := T$. We leave to the reader to check that such formalization is indeed possible. For part (b), notice that for every $x \in (\omega 2)^M \cap (\omega 2)^N$ the two combinatorial trees $f_T^M(x) = (\mathbb{G}_{\Sigma_T(x)})^M$ and $f_T^N(x) = (\mathbb{G}_{\Sigma_T(x)})^N$ must coincide because they are explicitly computed in ZF using just x , κ and T as parameters and all the bijections $\langle\langle \cdot \rangle\rangle$, $\langle \cdot, \cdot \rangle$, and e_u involved in their coding as structures on κ are absolute between transitive models of ZF. \square

REMARK 11.11. Since the transitive ZF-models M, N involved in its statements may be proper classes, Proposition 11.10 is actually a second-order statement. However, by restricting the kind of models on which M and N can vary, one can often reformulate such a result as a purely first-order statement — see Section 14.2 (and in particular Examples 14.17) for more on this. A similar remark holds also for the subsequent Proposition 11.13.

LEMMA 11.12. *Given an infinite cardinal κ and a tree $T \in \mathbb{T}_\kappa$ such that $R = p[T]$ is a quasi-order, let f_T be the map defined in (11.5). Then for arbitrary $x, y \in \omega 2$, if $f_T(x) \preceq f_T(y)$, then there is a (canonical) witness of this fact which is explicitly LST-definable ZF using only x , y , κ , and T as parameters.*

PROOF. If $f_T(x) \preceq f_T(y)$, then $x R y$ by Theorem 11.8(a). Pick the leftmost branch $b_{x,y}$ such that $(x, y, b_{x,y}) \in [\tilde{T}]$ (where \tilde{T} is the faithful representation of R constructed from T as in (11.3)), and apply (11.4) to $\xi := b_{x,y}$ to get a (canonical) $\varphi: {}^{<\omega}\kappa \rightarrow {}^{<\omega}\kappa$ witnessing $\Sigma_T(x) \preceq_{\max}^\kappa \Sigma_T(y)$. Then use this φ to define the desired canonical embedding i of $f_T(x)$ into $f_T(y)$ as in the first part of the proof of Theorem 11.8(a). \square

Given an infinite cardinal κ , the embeddability relation on $\text{Mod}_\mathcal{L}^\kappa$, being defined by a Σ_1 LST-formula, is upward absolute but in general not downward absolute: for example, for suitable

$X, Y \in \text{CT}_\kappa$ one may be able to add by forcing an embedding of X into Y (so that $X \sqsubset Y$ holds in a suitable generic extension) even if X does not embed into Y in the ground model. In contrast, we are now going to show that the embeddability relation on the range of an f_T as in (11.5) is always absolute for transitive models of ZF containing κ and T .

PROPOSITION 11.13. *Let M_0, M_1 be transitive models of ZF, and let κ and T be such that $\kappa \in \text{Card}^{M_i}$, $T \in (\mathbb{T}_\kappa)^{M_i}$, and $R^{M_i} := (p[T])^{M_i}$ is a quasi-order in M_i (for $i = 0, 1$). Recall that $f_T^{M_0}(z) = f_T^{M_1}(z)$ for all $z \in (\omega 2)^{M_0} \cap (\omega 2)^{M_1}$ by Proposition 11.10(b), so that we can unambiguously set $f_T(z) := f_T^{M_0}(z) = f_T^{M_1}(z)$ for all such z . Then for every $x, y \in (\omega 2)^{M_0} \cap (\omega 2)^{M_1}$*

$$M_0 \models f_T(x) \sqsubset f_T(y) \Leftrightarrow M_1 \models f_T(x) \sqsubset f_T(y).$$

PROOF. By Theorem 11.8(a) (which holds both in M_0 and M_1) and absoluteness of existence of infinite branches through descriptive set-theoretic trees on κ , we have that

$$\begin{aligned} M_0 \models f_T(x) \sqsubset f_T(y) &\Leftrightarrow M_0 \models \exists \xi \in {}^\omega \kappa ((x, y, \xi) \in [T]) \\ &\Leftrightarrow M_1 \models \exists \xi \in {}^\omega \kappa ((x, y, \xi) \in [T]) \\ &\Leftrightarrow M_1 \models f_T(x) \sqsubset f_T(y). \end{aligned}$$

□

12. Invariant universality

As mentioned in the introduction, the isomorphism relation \cong_σ^ω on the set Mod_σ^ω of countable models of an $\mathcal{L}_{\omega_1\omega}$ -sentence σ is an analytic equivalence relation, and these equivalence relations have been extensively studied in the literature (see [Vau75, BK96] and the references therein). In [LR05] the analytic quasi-order \sqsubset_σ^ω of embeddability between countable models of σ (where e.g. $\sigma := \sigma_{\text{CT}}$ is the $\mathcal{L}_{\omega_1\omega}$ -sentence axiomatizing combinatorial trees from page 73) has been shown to be \leq_B -complete for the class of analytic quasi-order. In [FMR11] a strengthening of Borel-completeness (called invariant universality in [CMMR13]) which bears on both \cong and \sqsubset has been introduced. Here this notion is generalized to arbitrary infinite cardinals.

DEFINITION 12.1. Let \mathcal{C} be a class of quasi-orders, \mathcal{L} be a finite relational language, and κ be an infinite cardinal. The embeddability relation $\sqsubset_{\mathcal{L}}^\kappa$ is **invariantly universal for \mathcal{C}** if for every $R \in \mathcal{C}$ there is an $\mathcal{L}_{\kappa^+\kappa}$ -sentence σ such that $R \sim \sqsubset_\sigma^\kappa$.

A localized version of invariant universality can be defined in a similar way.

DEFINITION 12.2. Let \mathcal{C} , \mathcal{L} , and κ be as in Definition 12.1. Given an $\mathcal{L}_{\kappa^+\kappa}$ -sentence τ , the embeddability relation \sqsubset_τ^κ is **invariantly universal for \mathcal{C}** if for every $R \in \mathcal{C}$ there is an $\mathcal{L}_{\kappa^+\kappa}$ -sentence σ such that $\text{Mod}_\sigma^\kappa \subseteq \text{Mod}_\tau^\kappa$ and $R \sim \sqsubset_\sigma^\kappa$.

As for the case of (\leq_*) -completeness, when in Definitions 12.1 and 12.2 the reducibility \leq is replaced by one of its restricted forms \leq_* we speak of **\leq_* -invariant universality**.

In order to establish the invariant universality of $\sqsubset_{\text{CT}}^\kappa$ for κ -Souslin quasi-orders on ${}^\omega 2$, in this section we will have to construct some infinitary sentences, denoted by Ψ and σ_T , and some maps between their sets of models of size κ : the reader is advised to refer to Figure 5 in order to keep track of these functions.

12.1. An $\mathcal{L}_{\kappa^+\kappa}$ -sentence Ψ describing the structures \mathbb{G}_S . Henceforth we fix an uncountable cardinal κ . We will now begin the definition of a sequence of nine $\mathcal{L}_{\kappa^+\kappa}$ -sentences Φ_0, \dots, Φ_8 which will be crucial for the proof of our main result. These sentences try to describe with some accuracy the common properties of the structures of the form \mathbb{G}_S for $S \in \text{Tr}(2 \times \kappa)$ defined in Section 10. To help in understanding the intended meaning of such sentences, we will freely use the following two conventions (besides the ones already explained in Section 8.1.1):

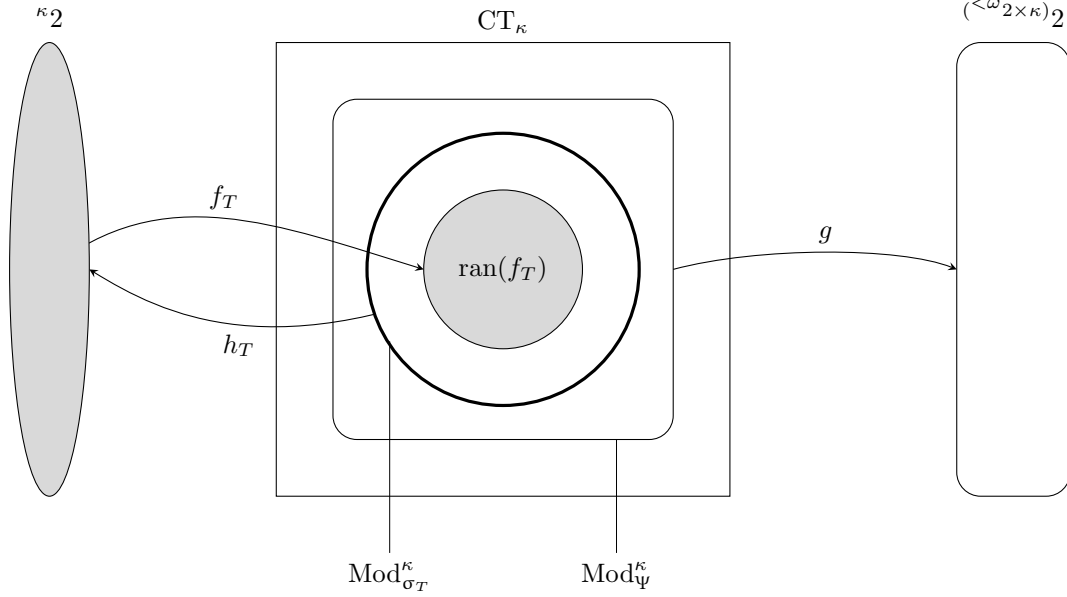


FIGURE 5. The reductions used in Section 12. The set $\text{Mod}_{\sigma_T}^\kappa$ is the saturation of $\text{ran}(f_T)$, where f_T is the map defined in (11.5).

- we will use metavariables $x, y, z, x_\alpha, y_\alpha, z_\alpha$ (possibly with various decorations or different subscripts) instead of the v_α 's;
- we will consider (infinitary) conjunctions and disjunctions over *sets* of formulæ of *size* $\leq \kappa$ (instead of conjunctions and disjunctions over *sequences of length* $< \kappa^+$, as Definition 8.1 would officially require), as long as it is clear that such sets can be well-ordered in a canonical way in ZF (usually by means of the coding functions $\langle\langle \cdot \rangle\rangle$, $\langle \cdot, \cdot \rangle$, and e_u for $u \in {}^{<\omega}2$ from (2.2), (2.1) and (10.6), respectively, which are absolute for transitive models of ZF).

This means that, formally, the $\mathcal{L}_{\kappa^+\kappa}$ -sentence Φ_i is obtained from the displayed one by substituting in the natural way each metavariable with a corresponding variable in the official list $\langle v_\alpha \mid \alpha < \kappa \rangle$, and by well-ordering in a canonical way all the sets of subformulæ to which an (infinitary) conjunction or disjunction is applied. We will explicitly perform this formalization just for the first few formulæ, leaving to the reader all other cases.

Given a variable x and $0 \neq n \in \omega$, let $d_{<n}(x)$ denote the $\mathcal{L}_{\omega\omega}$ -formula

$$(d_{<n}(x)) \quad \forall \langle x_i \mid i < n \rangle \left(\bigwedge_{i < n} x_i \text{ E } x \Rightarrow \bigvee_{i < j < n} x_i \simeq x_j \right)$$

(where if $n = 1$ we agree that $\bigvee_{i < j < n} x_i \simeq x_j$ is any inconsistent sentence), and abbreviate by $d_{\geq n}(x)$, $d_{=n}(x)$, and $d_{\neq n}(x)$ the $\mathcal{L}_{\omega\omega}$ -formulæ $\neg(d_{<n}(x))$, $d_{<n+1}(x) \wedge d_{\geq n}(x)$, and $\neg(d_{=n}(x))$, respectively. If a is a vertex of a graph X , then $X \models d_{<n}[a]$ if and only if a has degree $< n$ in X . To completely formalize the formula $d_{<n}(x)$, one should first fix $\alpha < \kappa$ and enumeration $\langle c_m \mid m < \frac{n(n-1)}{2} \rangle$ of all pairs $(i, j) \in \omega^2$ with $i < j < n$, and then define $d_{<n}(v_\alpha)$ as

$$\forall \langle v_{\alpha+i+1} \mid i < n \rangle \left(\bigwedge_{i < n} v_{\alpha+i+1} \text{ E } v_\alpha \Rightarrow \bigvee_{m < \frac{n(n-1)}{2}} v_{\alpha+i_m+1} \simeq v_{\alpha+j_m+1} \right),$$

where $(i_m, j_m) \in \omega^2$ is the unique pair such that $c_m = (i_m, j_m)$. A similar formalization may easily be obtained for the following auxiliary $\mathcal{L}_{\omega\omega}$ -formulae:

$$(\text{Seq}(x)) \quad d_{\geq 4}(x) \wedge \forall y (x \mathbf{E} y \Rightarrow d_{=2}(x));$$

$$(\text{Seq}^-(x)) \quad d_{=2}(x) \wedge \forall y (x \mathbf{E} y \Rightarrow \text{Seq}(y));$$

$$(\text{F}(x)) \quad d_{<4}(x) \wedge \forall y [x \mathbf{E} y \Rightarrow (d_{<4}(y) \vee \text{Seq}(y))] \wedge \exists z [x \mathbf{E} z \wedge d_{<4}(z)].$$

$$(\text{U}(x)) \quad d_{\neq 2}(x) \wedge \exists y, z [z \neq x \wedge x \mathbf{E} y \wedge y \mathbf{E} z \wedge d_{\geq 4}(y) \wedge d_{\geq 4}(z)];$$

$$(\widehat{\text{Seq}}(x)) \quad d_{=2}(x) \wedge \exists y [\text{U}(y) \wedge x \mathbf{E} y \wedge d_{\geq 4}(y)] \wedge \forall z [z \mathbf{E} x \Rightarrow d_{\geq 4}(z)];$$

REMARK 12.3. It is not hard to check that for every $S \in \text{Tr}(2 \times \kappa)$ and every vertex $a \in \mathbb{G}_S$ (where \mathbb{G}_S is defined as in Section 10.2) we have

$$\mathbb{G}_S \models \text{Seq}[a] \Leftrightarrow a \in \text{Seq}(\mathbb{G}_S),$$

and analogous results hold for the other formulae $\text{Seq}^-(x)$, $\text{U}(x)$, $\widehat{\text{Seq}}(x)$, and $\text{F}(x)$ and the corresponding substructures $\text{Seq}^-(\mathbb{G}_S)$, $\text{U}(\mathbb{G}_S)$, $\widehat{\text{Seq}}(\mathbb{G}_S)$, and $\text{F}(\mathbb{G}_S)$ of \mathbb{G}_S defined in (10.7a)–(10.7g).

We are now ready to introduce the first $\mathcal{L}_{\kappa+\kappa}$ -sentence Φ_0 (which is actually an $\mathcal{L}_{\omega_1\omega}$ -sentence):

$$(\Phi_0) \quad \varphi_{\text{CT}} \wedge \forall x [\text{Seq}(x) \vee \text{Seq}^-(x) \vee \widehat{\text{Seq}}(x) \vee \text{U}(x) \vee \text{F}(x)].$$

REMARK 12.4. Notice that the formulae appearing in the disjunction of Φ_0 are mutually exclusive: each element of an arbitrary \mathcal{L} -structure X satisfying Φ_0 can realize at most one of $\text{Seq}(x)$, $\text{Seq}^-(x)$, $\widehat{\text{Seq}}(x)$, $\text{U}(x)$, or $\text{F}(x)$.

In any combinatorial tree of the form \mathbb{G}_S (for $S \in \text{Tr}(2 \times \kappa)$) one has that for every vertex a in $\text{U}(\mathbb{G}_S)$ there is a unique $b \in \widehat{\text{Seq}}(\mathbb{G}_S)$ such that a is connected to b by a (finite) chain all of whose intermediate points are in $\text{U}(\mathbb{G}_S)$ as well. This property of a and b is rendered by the following $\mathcal{L}_{\omega_1\omega}$ -formula:

$$(\text{root}(y, x)) \quad \text{U}(y) \wedge \widehat{\text{Seq}}(x) \wedge \bigvee_{n < \omega} \exists \langle x_i \mid i \leq n \rangle \left[\bigwedge_{0 < i \leq n} \text{U}(x_i) \wedge x \simeq x_0 \wedge x_n \mathbf{E} y \wedge \bigwedge_{i < n} x_i \mathbf{E} x_{i+1} \right].$$

This allows us to write the $\mathcal{L}_{\omega_1\omega}$ -formula Φ_1 , which expresses the above mentioned property of the structures \mathbb{G}_S (the symbol $\exists!$ denotes the quantifier “there exists a unique”, which may be expressed using the other quantifiers and connectives in the usual way):

$$(\Phi_1) \quad \forall y [\text{U}(y) \Rightarrow \exists! x (\text{root}(y, x))].$$

Let now X be an arbitrary \mathcal{L} -structure of size $< \kappa$, and $i: X \rightarrow \kappa$ be any injection. We denote by

$$\tau_{\text{qf}}^i(X)(\langle v_\alpha \mid \alpha \in \text{ran}(i) \rangle)$$

the **quantifier-free type of X (induced by i)**, i.e. the $\mathcal{L}_{\kappa\kappa}^0$ -formula

$$\bigwedge_{\substack{x, y \in X \\ x \neq y}} (v_{i(x)} \neq v_{i(y)}) \wedge \bigwedge_{\substack{x, y \in X \\ x \mathbf{E}^X y}} (v_{i(x)} \mathbf{E} v_{i(y)}) \wedge \bigwedge_{\substack{x, y \in X \\ \neg(x \mathbf{E}^X y)}} \neg(v_{i(x)} \mathbf{E} v_{i(y)}).$$

To completely formalize this sentence, one of course need to well-order the infinitary conjunctions above using the given injection i . Notice that if Y is an \mathcal{L} -structure and $\langle a_\alpha \mid \alpha \in \text{ran}(i) \rangle$, $\langle b_\alpha \mid \alpha \in \text{ran}(i) \rangle$ are two sequences of elements of Y such that both $Y \models \tau_{\text{qf}}^i(X)[\langle a_\alpha \mid \alpha \in \text{ran}(i) \rangle]$ and $Y \models \tau_{\text{qf}}^i(X)[\langle b_\alpha \mid \alpha \in \text{ran}(i) \rangle]$, then $Y \restriction \{a_\alpha \mid \alpha \in \text{ran}(i)\}$ and $Y \restriction \{b_\alpha \mid \alpha \in \text{ran}(i)\}$ are isomorphic via the map $a_\alpha \mapsto b_\alpha$ (in fact, they are both isomorphic to X). In this paper, the previous procedure will be applied only to structures X which are *canonically well-orderable* in **ZF** using the coding maps $\langle\langle \cdot \rangle\rangle$, $\langle \cdot, \cdot \rangle$, and e_u for $u \in {}^{<\omega}2$ in the obvious way — in fact, the domains of these structures will in general be subsets of size $< \kappa$ of ${}^{<\omega}\kappa$. In all such cases we will thus have a canonical injection $i = i_X: X \rightarrow \kappa$ (namely, the one induced by the coding map $\langle\langle \cdot \rangle\rangle$): in order to simplify the notation, we will then safely drop the reference to such i , replace variables with metavariables, and call the resulting expression **qf-type of X** . In the formulæ below we will denote the qf-type of such an \mathcal{L} -structure X simply by

$$\tau_{\text{qf}}(X)(\langle x_i \mid i \in X \rangle).$$

The next $\mathcal{L}_{\kappa+\kappa}$ -sentences Φ_2 , Φ_3 , and Φ_4 will complete the description of the substructure \mathbb{G}_1 of any \mathbb{G}_S (see Lemma 12.5). To this aim, for each $s \in {}^{<\omega}\kappa$ we first introduce the following auxiliary $\mathcal{L}_{\kappa+\kappa}$ -formulæ:

$$\begin{aligned} (\widehat{\text{Seq}}_s(x)) \quad & \widehat{\text{Seq}}(x) \wedge \exists \langle x_i \mid i \in U_{\langle\langle s \rangle\rangle} \rangle \left[\bigwedge_{i \in U_{\langle\langle s \rangle\rangle}} \text{root}(x_i, x) \wedge \forall y (\text{root}(y, x) \Rightarrow \bigvee_{i \in U_{\langle\langle s \rangle\rangle}} y \simeq x_i) \right. \\ & \left. \wedge x \mathbf{E} x_\emptyset \wedge \left(\bigwedge_{\emptyset \neq i \in U_{\langle\langle s \rangle\rangle}} \neg(x \mathbf{E} x_i) \right) \wedge \tau_{\text{qf}}(U_{\langle\langle s \rangle\rangle})(\langle x_i \mid i \in U_{\langle\langle s \rangle\rangle} \rangle) \right]; \end{aligned}$$

$$(\text{Seq}_s(x)) \quad \text{Seq}(x) \wedge \exists y (\widehat{\text{Seq}}_s(y) \wedge x \mathbf{E} y);$$

$$(\text{Seq}_s^-(x)) \quad \text{Seq}^-(x) \wedge \exists y, z (\text{Seq}_{s^*}(y) \wedge \text{Seq}_s(z) \wedge y \mathbf{E} x \wedge x \mathbf{E} z).$$

Recall from (2.4) that for $\emptyset \neq s \in {}^{<\omega}\kappa$ we set $s^* = s \restriction (\text{lh } s - 1)$, so $\text{Seq}_s^-(x)$ is defined only for $s \neq \emptyset$.

Notice that if $S \in \text{Tr}(2 \times \kappa)$ and $a \in \mathbb{G}_S$, then

$$\mathbb{G}_S \models \widehat{\text{Seq}}_s[a] \Leftrightarrow a = \widehat{s}.$$

Similarly,

$$\mathbb{G}_S \models \text{Seq}_s[a] \Leftrightarrow a = s \quad \text{and} \quad \mathbb{G}_S \models \text{Seq}_s^-[a] \Leftrightarrow a = s^-.$$

Now let

$$(\Phi_2) \quad \forall x \left[\widehat{\text{Seq}}(x) \Rightarrow \bigvee_{s \in {}^{<\omega}\kappa} \widehat{\text{Seq}}_s(x) \right] \wedge \bigwedge_{s \in {}^{<\omega}\kappa} \exists! x (\widehat{\text{Seq}}_s(x));$$

$$(\Phi_3) \quad \forall x [\text{Seq}(x) \Rightarrow \exists! y (\widehat{\text{Seq}}(y) \wedge x \mathbf{E} y)] \wedge \bigwedge_{s \in {}^{<\omega}\kappa} \exists! x \text{Seq}_s(x);$$

$$(\Phi_4) \quad \forall x \left[\text{Seq}^-(x) \Rightarrow \bigvee_{\emptyset \neq s \in {}^{<\omega}\kappa} \text{Seq}_s^-(x) \right] \wedge \bigwedge_{\emptyset \neq s \in {}^{<\omega}\kappa} \exists! x (\text{Seq}_s^-(x)).$$

It is not hard to see that $\mathbb{G}_S \models \bigwedge_{i \leq 4} \Phi_i$ for every $S \in \text{Tr}(2 \times \omega)$. We are now going to show in Lemma 12.5 that every model of $\bigwedge_{i \leq 4} \Phi_i$ contains a substructure (canonically) isomorphic to \mathbb{G}_1 . Let us first fix some notation. For $X \in \text{Mod}_{\mathcal{L}}^\kappa$, set

$$(12.1a) \quad \text{Seq}(X) := \{a \in X \mid X \models \text{Seq}[a]\}$$

$$(12.1b) \quad \text{Seq}^-(X) := \{a \in X \mid X \models \text{Seq}^-[a]\}$$

$$(12.1c) \quad \widehat{\text{Seq}}(X) := \{a \in X \mid X \models \widehat{\text{Seq}}[a]\}$$

$$(12.1d) \quad \text{U}(X) := \{a \in X \mid X \models \text{U}[a]\}$$

$$(12.1e) \quad \text{F}(X) := \{a \in X \mid X \models \text{F}[a]\}$$

$$(12.1f) \quad \mathbb{G}_0(X) := \text{Seq}(X) \cup \text{Seq}^-(X)$$

$$(12.1g) \quad \mathbb{G}_1(X) := \mathbb{G}_0(X) \cup \widehat{\text{Seq}}(X) \cup \text{U}(X).$$

(Notice that when $X = \mathbb{G}_S$ for some $S \in \text{Tr}(2 \times \kappa)$ this notation is coherent with the one established in (10.7a)–(10.7c), (10.7e) and (10.7g), and that the unions in the definition of $\mathbb{G}_0(X)$ and $\mathbb{G}_1(X)$ are necessarily disjoint by Remark 12.4.)

LEMMA 12.5. *For every $X \in \text{CT}_\kappa$, if $X \models \bigwedge_{i \leq 4} \Phi_i$, then $\mathbb{G}_1(X) \cong \mathbb{G}_1$ via a canonical isomorphism $i_X: \mathbb{G}_1(X) \rightarrow \mathbb{G}_1$.*

PROOF. Recall the \mathcal{L} -structures Seq , Seq^- , $\widehat{\text{Seq}}$, U_s , and U defined in Remark 10.7. We will canonically define some partial isomorphisms

$$\iota_{\widehat{\text{Seq}}}: \widehat{\text{Seq}}(X) \rightarrow \widehat{\text{Seq}}$$

$$\iota_{\text{U}}: \text{U}(X) \rightarrow \text{U}$$

$$\iota_{\text{Seq}}: \text{Seq}(X) \rightarrow \text{Seq}$$

$$\iota_{\text{Seq}^-}: \text{Seq}^-(X) \rightarrow \text{Seq}^-.$$

and then show that the union ι_X of these maps is the desired canonical isomorphism.

By $X \models \Phi_2$, we get that there is a bijection $j: \widehat{\text{Seq}}(X) \rightarrow {}^{<\omega}\kappa$ (namely, the map sending $a \in \widehat{\text{Seq}}(X)$ to the unique $s \in {}^{<\omega}\kappa$ such that $X \models \widehat{\text{Seq}}_s[a]$), so we can define the bijection $\iota_{\widehat{\text{Seq}}}: \widehat{\text{Seq}}(X) \rightarrow \widehat{\text{Seq}}: a \mapsto j(a)$.

For each $s \in {}^{<\omega}\kappa$, let

$$\text{U}_s(X) := \{a \in X \mid X \models \text{root}[a, j^{-1}(s)]\}.$$

(Notice that this notation is again coherent with (10.7d) when $X = \mathbb{G}_S$ for some $S \in \text{Tr}(2 \times \kappa)$.) Then $\{\text{U}_s(X) \mid s \in {}^{<\omega}\kappa\}$ is a partition of $\text{U}(X)$ by $X \models \Phi_1$. Moreover, using $X \models \widehat{\text{Seq}}_s[j^{-1}(s)]$ we have that each $\text{U}_s(X)$ is isomorphic to U_s via some canonical ι_s which maps the unique vertex in $\text{U}_s(X)$ adjacent to $j^{-1}(s)$ to the point (s, \emptyset) . (To see that we do not need any choice to pick the isomorphisms ι_s , use Lemma 10.5(f) to first get a canonical isomorphism $i_{\text{U}_s(X), \langle\langle s \rangle\rangle}: \text{U}_s(X) \rightarrow \text{U}_{\langle\langle s \rangle\rangle}$, notice that by Lemma 10.5(e) we can always assume that the map $i_{\text{U}_s(X), \langle\langle s \rangle\rangle}$ sends the unique vertex in $\text{U}_s(X)$ adjacent to $j^{-1}(s)$ to \emptyset , and then set $\iota_s(a) := (s, i_{\text{U}_s(X), \langle\langle s \rangle\rangle}(a))$ for every $a \in \text{U}_s(X)$.) Set $\iota_{\text{U}} := \bigcup_{s \in {}^{<\omega}\kappa} \iota_s$, and notice that it is a well-defined bijection between $\text{U}(X)$ and U because the $\text{U}_s(X)$ are pairwise disjoint.

CLAIM 12.5.1. $\iota' := \iota_{\widehat{\text{Seq}}} \cup \iota_{\text{U}}$ is a partial isomorphism between $\mathbb{G}_1(X) \setminus \mathbb{G}_0(X) = \widehat{\text{Seq}}(X) \cup \text{U}(X)$ and $\mathbb{G}_1 \setminus \mathbb{G}_0 = \widehat{\text{Seq}} \cup \text{U}$.

PROOF OF THE CLAIM. The fact that ι' is a bijection between $\mathbb{G}_1(X) \setminus \mathbb{G}_0(X)$ and $\mathbb{G}_1 \setminus \mathbb{G}_0$ is obvious, so we just need to show that it preserves the edge relation. Let $a, b \in \widehat{\text{Seq}}(X) \cup U(X)$. If $X \models \widehat{\text{Seq}}[a]$, then all neighborhoods of a must have degree ≥ 4 , so if $a \mathbf{E}^X b$, then $X \not\models \widehat{\text{Seq}}[b]$: this shows that any two points in $\widehat{\text{Seq}}(X)$ are not connected by an edge.

Suppose now $a, b \in U(X)$. If $a \in U_s(X)$ and $b \in U_t(X)$ for distinct $s, t \in {}^{<\omega}\kappa$, then $\neg(a \mathbf{E}^X b)$, because otherwise $X \models \text{root}[a, j^{-1}(s)] \wedge \text{root}[a, j^{-1}(t)]$, contradicting $X \models \Phi_1$. If instead $a, b \in U_s(X)$ for the same $s \in {}^{<\omega}\kappa$, then $\iota'(a) = \iota_s(a) \in U_s$ and $\iota'(b) = \iota_s(b) \in U_s$, so

$$a \mathbf{E}^X b \Leftrightarrow \iota'(a) \mathbf{E}^{U_s} \iota'(b) \Leftrightarrow \iota'(a) \mathbf{E}^{\mathbb{G}_1} \iota'(b)$$

by the choice of ι_s and $\mathbb{G}_1 \upharpoonright U_s = U_s$.

Finally, assume that $a \in \widehat{\text{Seq}}(X)$ and $b \in U(X)$, and let $s \in {}^{<\omega}\kappa$ be such that $j(a) = s$ (so that $\iota'(a) = \iota_{\widehat{\text{Seq}}}(a) = \hat{s}$). If $b \in U_t(X)$ for some $s \neq t \in {}^{<\omega}\kappa$, then $\neg(a \mathbf{E}^X b)$, as otherwise $X \models \text{root}[b, j^{-1}(s)] \wedge \text{root}[b, j^{-1}(t)]$, contradicting $X \models \Phi_1$ again. If instead $b \in U_s(X)$, then $a \mathbf{E}^X b \Leftrightarrow \iota'(b) = \iota_s(b) = (s, \emptyset)$ by our choice of ι_s , so that $a \mathbf{E}^X b \Leftrightarrow \iota'(a) \mathbf{E}^{\mathbb{G}_1} \iota'(b)$ by Definition 10.4.

By checking the definition of $\mathbf{E}^{\mathbb{G}_1}$ in Definition 10.4, the previous observations suffice to show the desired result. \square

Let us now consider an arbitrary point $a \in \text{Seq}(X)$. Since $X \models \Phi_3$, there is a unique $b \in \widehat{\text{Seq}}(X)$ with $b \mathbf{E}^X a$: therefore, we can unambiguously set $\iota_{\text{Seq}}(a) := j(b)$, and check that by $X \models \Phi_3$ the map $\iota_{\text{Seq}}(X) : \text{Seq}(X) \rightarrow \text{Seq}$ is a bijection. Moreover, for every $a \in X$ and $s \in {}^{<\omega}\kappa$ we have $X \models \text{Seq}_s[a] \Leftrightarrow \iota_{\text{Seq}}(a) = s$.

Finally, if $a \in \text{Seq}^-(X)$, then by $X \models \Phi_4$ there is a unique $\emptyset \neq s \in {}^{<\omega}\kappa$ such that $X \models \text{Seq}_s^-[a]$, so that the map ι_{Seq^-} sending a to $\iota_{\text{Seq}^-}(a) := s^-$ is a bijection between $\text{Seq}^-(X)$ and Seq^- .

Consider now the canonical bijection $\iota_X : \mathbb{G}_1(X) \rightarrow \mathbb{G}_1$ defined by

$$(12.2) \quad \iota_X := \iota_{\text{Seq}} \cup \iota_{\text{Seq}^-} \cup \iota_{\widehat{\text{Seq}}} \cup \iota_U,$$

which is well-defined since the functions appearing in the union have pairwise disjoint domains. We claim that ι_X is an isomorphism between $\mathbb{G}_1(X)$ and \mathbb{G}_1 , so let us fix arbitrary $a, b \in \mathbb{G}_1(X)$. Since by Claim 12.5.1 we already know that $\iota_X \upharpoonright (\mathbb{G}_1(X) \setminus \mathbb{G}_0(X)) = \iota'$ is an isomorphism between $\mathbb{G}_1(X) \setminus \mathbb{G}_0(X)$ and $\mathbb{G}_1 \setminus \mathbb{G}_0$, we may assume without loss of generality that $a \in \mathbb{G}_0(X)$. Suppose first that $b \in \mathbb{G}_1(X) \setminus \mathbb{G}_0(X)$. If $a \in \text{Seq}^-(X)$, then $\neg(a \mathbf{E}^X b)$ because a has only two neighbors each of which must be in $\text{Seq}(X)$ (by $X \models \text{Seq}^-[a]$), while $b \notin \mathbb{G}_0(X) \supseteq \widehat{\text{Seq}}(X)$ by case assumption. If instead $a \in \text{Seq}(X)$ and $b \in \widehat{\text{Seq}}(X)$, then $a \mathbf{E}^X b \Leftrightarrow \iota_X(b) = \iota_X(a)$ by definition of ι_{Seq} and $\iota_{\widehat{\text{Seq}}}$. Finally, if $a \in \text{Seq}(X)$ and $b \in U(X)$, then $\neg(a \mathbf{E}^X b)$ because a must have only neighbors of degree 2 by $X \models \text{Seq}[a]$, while b has degree $\neq 2$ in X by $X \models U[b]$.

Suppose now that $a, b \in \mathbb{G}_0(X)$, and let us assume that in fact $a \in \text{Seq}^-(X)$. Then $X \models \text{Seq}_s^-[a]$ for some $\emptyset \neq s \in {}^{<\omega}\kappa$, and $\iota_X(a) = \iota_{\text{Seq}^-}(a) = s^-$ by definition of ι_{Seq^-} . This implies that a has only two neighbors c_0, c_1 in X , and they are such that $X \models \text{Seq}_{s^*}[c_0] \wedge \text{Seq}_s[c_1]$: therefore $\iota_X(c_0) = \iota_{\text{Seq}}(c_0) = s^*$ and $\iota_X(c_1) = \iota_{\text{Seq}}(c_1) = s$. It follows that b is connected by an edge to a if and only if $b = c_0 \vee b = c_1$ if and only if $\iota_X(b) = s^* \vee \iota_X(b) = s$. The same argument (with a and b switched) takes care of the case $b \in \text{Seq}^-(X)$, so we just need to consider the case $a, b \in \text{Seq}(X)$. But then $\neg(a \mathbf{E}^X b)$ because b has degree ≥ 4 by $X \models \text{Seq}[b]$, while all neighbors of a must have degree 2 by $X \models \text{Seq}[a]$.

Checking the definition of $\mathbf{E}^{\mathbb{G}_1}$ in Definition 10.4 again, it is now easy to check that the above observations suffice to show that $\iota_X : \mathbb{G}_1(X) \rightarrow \mathbb{G}_1$ is an isomorphism. \square

Given $u \in {}^{<\omega}2$ and a sequence $\langle x_i \mid i \in F_u \rangle$ of variables, let $F_u(\langle x_i \mid i \in F_u \rangle)$ abbreviate the following $\mathcal{L}_{\omega_1\omega}$ -formula:

$$(F_u(\langle x_i \mid i \in F_u \rangle)) \quad \left(\bigwedge_{\emptyset \neq i \in F_u} F(x_i) \right) \wedge \text{Seq}(x_\emptyset) \wedge \tau_{\text{qf}}(F_u)(\langle x_i \mid i \in F_u \rangle).$$

Given $X \in \text{Mod}_{\mathcal{L}}^\kappa$ and $u \in {}^{<\omega}2$, we call X -**fork (coding u)** any substructure of X determined by a sequence of points $\langle a_i \in X \mid i \in F_u \rangle$ such that $X \models F_u[\langle a_i \mid i \in F_u \rangle]$, and the point a_\emptyset is called **root** of such X -fork.

We now provide $\mathcal{L}_{\omega_1\omega_1}$ -sentences Φ_5, \dots, Φ_8 which, together with the previous ones Φ_0, \dots, Φ_4 , complete the description of an \mathcal{L} -structure of the form \mathbb{G}_S (for any $S \in \text{Tr}(2 \times \omega)$):

$$(\Phi_5) \quad \forall x \left[F(x) \Rightarrow \bigvee_{u \in {}^{<\omega}2} \exists \langle x_i \mid i \in F_u \rangle \left(F_u(\langle x_i \mid i \in F_u \rangle) \wedge \bigvee_{\emptyset \neq i \in F_u} x \simeq x_i \right) \right];$$

$$(\Phi_6) \quad \forall x \bigwedge_{u \in {}^{<\omega}2} \forall \langle y_i \mid i \in F_u \rangle \left[F_u(\langle y_i \mid i \in F_u \rangle) \wedge \bigwedge_{i \in F_u} x \neq y_i \right. \\ \left. \Rightarrow \bigwedge_{\emptyset \neq i \in F_u} \left(\neg(x \mathbf{E} y_i) \wedge \neg(y_i \mathbf{E} x) \right) \right];$$

$$(\Phi_7) \quad \bigwedge_{\substack{u \in {}^{<\omega}2 \\ v \in {}^{<\omega}2}} \forall \langle x_i \mid i \in F_u \rangle \forall \langle y_j \mid j \in F_v \rangle \left[F_u(\langle x_i \mid i \in F_u \rangle) \wedge F_v(\langle y_j \mid j \in F_v \rangle) \right. \\ \left. \Rightarrow \bigwedge_{\substack{\emptyset \neq i \in F_u \\ \emptyset \neq j \in F_v}} (x_i \neq y_j) \vee \left(\left(\bigwedge_{i \in F_u} \bigvee_{j \in F_v} (x_i \simeq y_j) \right) \wedge \left(\bigwedge_{j \in F_v} \bigvee_{i \in F_u} (y_j \simeq x_i) \right) \right) \right];$$

$$(\Phi_8) \quad \bigwedge_{u \in {}^{<\omega}2} \forall \langle x_i \mid i \in F_u \rangle \forall \langle y_j \mid j \in F_u \rangle \left[F_u(\langle x_i \mid i \in F_u \rangle) \wedge F_u(\langle y_j \mid j \in F_u \rangle) \right. \\ \left. \Rightarrow \left(\bigwedge_{i,j \in F_u} (x_i \neq y_j) \right) \vee \left(\bigwedge_{i \in F_u} (x_i \simeq y_i) \right) \right].$$

To understand the meaning of the above sentences, observe that for every $X \in \text{Mod}_{\mathcal{L}}^\kappa$ we have:

- $X \models \Phi_5$ if and only if every $a \in F(X)$ belongs to an X -fork (coding some $u \in {}^{<\omega}2$);
- $X \models \Phi_6$ if and only if given an arbitrary X -fork, if a point $a \in X$ does not belong to that fork then it can be connected by an edge only to its root;
- $X \models \Phi_7$ if and only if any two distinct X -forks may share only their roots;
- $X \models \Phi_8$ if and only if any two distinct X -forks coding *the same* $u \in {}^{<\omega}2$ must be disjoint.

Finally, let Ψ be the $\mathcal{L}_{\kappa+\kappa}$ -sentence

$$(12.3) \quad \bigwedge_{0 \leq i \leq 8} \Phi_i.$$

The following lemma is straightforward (see also Remark 12.3).

LEMMA 12.6. *Let $T \in \mathbb{T}_\kappa$, f_T be as in (11.5), and Ψ be the $\mathcal{L}_{\kappa+\kappa}$ -sentence in (12.3). Then $f_T(x) \models \Psi$ for every $x \in {}^\omega 2$, whence $\text{ran}(f_T) \subseteq \text{Mod}_{\Psi}^\kappa$ (see Figure 5).*

12.2. A classification of the structures in Mod_Ψ^κ up to isomorphism. Define the map

$$(12.4) \quad g: \text{Mod}_\Psi^\kappa \rightarrow {}^{<\omega}2^{\times \kappa 2}$$

(see Figure 5) as follows: given $X \in \text{Mod}_\Psi^\kappa$ and $(u, \alpha) \in {}^{<\omega}2 \times \kappa$, set $g(X)(u, \alpha) = 1$ if and only if there is an X -fork coding u whose root $a_\emptyset \in X$ is such that $X \models \text{Seq}_s[a_\emptyset]$ for the unique $s := s_\alpha \in {}^{<\omega}\kappa$ with $\langle\langle s \rangle\rangle = \alpha$, that is:

$$(12.5) \quad g(X)(u, \alpha) = 1 \Leftrightarrow X \models \exists \langle x_i \mid i \in F_u \rangle [F_u(\langle x_i \mid i \in F_u \rangle) \wedge \text{Seq}_{s_\alpha}(x_\emptyset)].$$

PROPOSITION 12.7. *The map g from (12.4) reduces \cong to $=$.*

PROOF. Let $X, Y \in \text{Mod}_\Psi^\kappa$ be isomorphic via some map ι , fix $(u, \alpha) \in {}^{<\omega}2 \times \kappa$, and let $s = s_\alpha \in {}^{<\omega}\kappa$ be such that $\langle\langle s \rangle\rangle = \alpha$. Then for every sequence $\langle a_i \mid i \in F_u \rangle$

$$X \models F_u[\langle a_i \mid i \in F_u \rangle] \wedge \text{Seq}_s[a_\emptyset] \Leftrightarrow Y \models F_u[\langle \iota(a_i) \mid i \in F_u \rangle] \wedge \text{Seq}_s[\iota(a_\emptyset)].$$

It follows that $g(X) = g(Y)$ by (12.5).

Conversely, let $X, Y \in \text{Mod}_\Psi^\kappa$, so that they both satisfy Φ_i for $0 \leq i \leq 8$, and assume that $g(X) = g(Y)$. By Lemma 12.5, there are canonical isomorphisms $\iota_X: \mathbb{G}_1(X) \rightarrow \mathbb{G}_1$ and $\iota_Y: \mathbb{G}_1(Y) \rightarrow \mathbb{G}_1$, where $\mathbb{G}_1(X)$ and $\mathbb{G}_1(Y)$ are defined as in (12.1g) and ι_X, ι_Y are the maps from (12.2). We will now extend the isomorphism $\iota_Y^{-1} \circ \iota_X: \mathbb{G}_1(X) \rightarrow \mathbb{G}_1(Y)$ to an isomorphism

$$\iota_{X,Y}: X \rightarrow Y.$$

Let $a \in X \setminus \mathbb{G}_1(X)$. By $X \models \Phi_0$ we have $X \models F[a]$, and hence by $X \models \Phi_5$ there are $u \in {}^{<\omega}2$ and $\langle a_i \mid i \in F_u \rangle$ such that $X \models F_u[\langle a_i \mid i \in F_u \rangle]$ and $a = a_{\bar{i}}$ for some $\emptyset \neq \bar{i} \in F_u$. In particular, $X \models \text{Seq}[a_\emptyset]$, so that by $X \models \Phi_2 \wedge \Phi_3$ there is (a unique) $s \in {}^{<\omega}\kappa$ such that $X \models \text{Seq}_s[a_\emptyset]$ (see also the definition of ι_{Seq} in the proof of Lemma 12.5). By definition of g we have $g(X)(u, \langle\langle s \rangle\rangle) = 1$, so $g(Y)(u, \langle\langle s \rangle\rangle) = 1$ by $g(X) = g(Y)$. Let $\langle b_i \mid i \in F_u \rangle$ be a sequence of elements of Y such that $Y \models F_u[\langle b_i \mid i \in F_u \rangle] \wedge \text{Seq}_s[b_\emptyset]$, and set $\iota_{X,Y}(a) := b_{\bar{i}}$. The definition of $\iota_{X,Y}(a)$ seems to depend on the choice of the sequence $\langle b_i \mid i \in F_u \rangle$, but the next claim shows that this is not the case.

CLAIM 12.7.1. *Suppose $s \in {}^{<\omega}\kappa$, $u \in {}^{<\omega}2$, and $\langle a_i \mid i \in F_u \rangle$ is a sequence of elements of X such $X \models F_u[\langle a_i \mid i \in F_u \rangle] \wedge \text{Seq}_s[a_\emptyset]$. Then there is a unique sequence $\langle b_i \mid i \in F_u \rangle$ of elements of Y such that $Y \models F_u[\langle b_i \mid i \in F_u \rangle] \wedge \text{Seq}_s[b_\emptyset]$. Therefore $\iota_{X,Y}(a_i) = b_i$ for every $i \in F_u$.*

PROOF OF THE CLAIM. Given two sequences $\langle b_i \mid i \in F_u \rangle$ and $\langle b'_i \mid i \in F_u \rangle$ as above, then $b_\emptyset = b'_\emptyset$ by $Y \models \text{Seq}_s[b_\emptyset] \wedge \text{Seq}_s[b'_\emptyset]$ and $Y \models \Phi_3$. This, together with $Y \models F_u[\langle b_i \mid i \in F_u \rangle]$ and $Y \models F_u[\langle b'_i \mid i \in F_u \rangle]$, implies $b_i = b'_i$ for all $i \in F_u$ by $Y \models \Phi_8$. \square

This shows that $\iota_{X,Y}: X \rightarrow Y$ is a well-defined map. We next check that it is a bijection.

CLAIM 12.7.2. *$\iota_{X,Y}$ is injective.*

PROOF OF THE CLAIM. Let $a, a' \in X$ be distinct. If at least one of a, a' belongs to $\mathbb{G}_1(X)$, then $\iota_{X,Y}(a) \neq \iota_{X,Y}(a')$ because $\iota_{X,Y} \upharpoonright \mathbb{G}_1(X) = \iota_Y^{-1} \circ \iota_X$ is a bijection between $\mathbb{G}_1(X)$ and $\mathbb{G}_1(Y)$, and $\iota_{X,Y}(X \setminus \mathbb{G}_1(X)) \subseteq Y \setminus \mathbb{G}_1(Y)$ by construction.

Thus we can assume $a, a' \notin \mathbb{G}_1(X)$, so that $X \models F[a]$ and $X \models F[a']$ by $X \models \Phi_0$. Assume towards a contradiction that $\iota_{X,Y}(a) = \iota_{X,Y}(a')$. Since $X \models \Phi_5$, there are $u, v \in {}^{<\omega}2$ and sequences $\langle a_i \mid i \in F_u \rangle$ and $\langle a'_i \mid i \in F_v \rangle$ such that $X \models F_u[\langle a_i \mid i \in F_u \rangle]$, $X \models F_v[\langle a'_i \mid i \in F_v \rangle]$, and $a = a_{\bar{i}}$, $a' = a'_{\bar{j}}$ for suitable $\emptyset \neq \bar{i} \in F_u$ and $\emptyset \neq \bar{j} \in F_v$. Moreover, by $X \models \Phi_2 \wedge \Phi_3$ there are $s, t \in {}^{<\omega}\kappa$ such that $X \models \text{Seq}_s[a_\emptyset]$ and $X \models \text{Seq}_t[a'_\emptyset]$. Since $g(X) = g(Y)$ there are two sequences $\langle b_i \mid i \in F_u \rangle$ and $\langle b'_j \mid j \in F_v \rangle$ such that $Y \models F_u[\langle b_i \mid i \in F_u \rangle] \wedge \text{Seq}_s[b_\emptyset]$ and $Y \models F_v[\langle b'_j \mid j \in F_v \rangle] \wedge \text{Seq}_t[b'_\emptyset]$ (and moreover such sequences are unique by Claim 12.7.1). Then

by definition of $\iota_{X,Y}$ we get $\iota_{X,Y}(a_i) = b_i$ and $\iota_{X,Y}(a'_j) = b'_j$ for every $i \in F_u$ and $j \in F_v$, so that in particular $\iota_{X,Y}(a) = b_{\bar{i}}$ and $\iota_{X,Y}(a') = b'_{\bar{j}}$. Since $Y \models \Phi_7$ and

$$(12.6) \quad b_{\bar{i}} = \iota_{X,Y}(a) = \iota_{X,Y}(a') = b'_{\bar{j}},$$

it follows that

$$(12.7) \quad \{b_i \mid i \in F_u\} = \{b'_j \mid j \in F_v\},$$

Since $Y \models \tau_{\text{qr}}(F_u)[\langle b_i \mid i \in F_u \rangle] \wedge \tau_{\text{qr}}(F_v)[\langle b'_j \mid j \in F_v \rangle]$, the substructure of Y with domain $\{b_i \mid i \in F_u\} = \{b'_j \mid j \in F_v\}$ is isomorphic to F_u via the map $b_i \mapsto i$ and to F_v via the map $b'_j \mapsto j$, so that $F_u \cong F_v$. Thus $u = v$ by (10.5c), and by (12.6) and $Y \models \Phi_8$ we get that $b_i = b'_i$ for all $i \in F_u = F_v$. Notice that this fact, together with (12.6) and the fact that all the b_i 's are necessarily distinct, also implies $\bar{i} = \bar{j}$. Moreover, $b_\emptyset = b'_\emptyset$. Since $a_\emptyset, a'_\emptyset \in \mathbb{G}_1(X)$, $\iota_{X,Y}(a_\emptyset) = b_\emptyset$, $\iota_{X,Y}(a'_\emptyset) = b'_\emptyset$, and $\iota_{X,Y} \upharpoonright \mathbb{G}_1(X) = \iota_Y^{-1} \circ \iota_X$ is a bijection, we get $a_\emptyset = a'_\emptyset$. Since $X \models \Phi_8$ and $u = v$, we then get $a_i = a'_i$ for all $i \in F_u = F_v$, and recalling that we showed $\bar{i} = \bar{j}$ we finally get $a = a_{\bar{i}} = a'_{\bar{i}} = a'_{\bar{j}} = a'$, a contradiction. \square

CLAIM 12.7.3. $\iota_{X,Y}$ is surjective.

PROOF OF THE CLAIM. Given $b \in Y$ we want to find $a \in X$ with $\iota_{X,Y}(a) = b$. If $b \in \mathbb{G}_1(Y)$, then this follows from the fact that $\iota_{X,Y} \upharpoonright \mathbb{G}_1(X)$ is a bijection between $\mathbb{G}_1(X)$ and $\mathbb{G}_1(Y)$, so we may assume $b \in Y \setminus \mathbb{G}_1(Y)$. Since $Y \models \Phi_0 \wedge \Phi_5$, this implies that there is $u \in {}^{<\omega}2$ and a sequence $\langle b_i \mid i \in F_u \rangle$ of elements of Y such that $Y \models F_u[\langle b_i \mid i \in F_u \rangle]$ and $b = b_{\bar{i}}$ for some $\emptyset \neq \bar{i} \in F_u$. Moreover, by $Y \models \Phi_2 \wedge \Phi_3$ we know that there is also $s \in {}^{<\omega}\kappa$ such that $Y \models \text{Seq}_s[b_\emptyset]$, so $g(Y)(u, \langle\langle s \rangle\rangle) = 1$. Since we are assuming $g(X) = g(Y)$, this means that there is a sequence $\langle a_i \mid i \in F_u \rangle$ of elements of X such that $X \models F_u[\langle a_i \mid i \in F_u \rangle] \wedge \text{Seq}_s[a_\emptyset]$. By our definition of $\iota_{X,Y}$ and Claim 12.7.1 we then have $\iota_{X,Y}(a_i) = b_i$. \square

It remains to show that $\iota_{X,Y}$ is also an isomorphism, i.e. that it preserves the edge relation. Fix $a, a' \in X$. Since $\iota_{X,Y} \upharpoonright \mathbb{G}_1(X) = \iota_Y^{-1} \circ \iota_X$ is already an isomorphism between $\mathbb{G}_1(X)$ and $\mathbb{G}_1(Y)$, we may assume without loss of generality that $a \notin \mathbb{G}_1(X)$. By the usual argument repeatedly used above, we then get from $g(X) = g(Y)$ that there are $u \in {}^{<\omega}2$, $s \in {}^{<\omega}\kappa$, a sequence $\langle a_i \mid i \in F_u \rangle$ of points of X , and $\bar{i} \in F_u$ such that $X \models F_u[\langle a_i \mid i \in F_u \rangle] \wedge \text{Seq}_s[a_\emptyset]$ and $a = a_{\bar{i}}$, together with a (unique) sequence of points $\langle b_i \mid i \in F_u \rangle$ of Y such that $Y \models F_u[\langle b_i \mid i \in F_u \rangle] \wedge \text{Seq}_s[b_\emptyset]$, so that $\iota_{X,Y}(a_i) = b_i$ for every $i \in F_u$ by construction (in particular, $\iota_{X,Y}(a) = b_{\bar{i}}$). We distinguish two cases. If $a' \neq a_i$ for every $i \in F_u$, then $\iota_{X,Y}(a') \neq b_i$ for every $i \in F_u$ by injectivity of $\iota_{X,Y}$ and the fact that $\iota_{X,Y}(a_i) = b_i$ for all $i \in F_u$. But since both X and Y satisfy Φ_6 , this implies that $a = a_{\bar{i}}$ is not \mathbf{E}^X -related to a' and $\iota_{X,Y}(a) = b_{\bar{i}}$ is not \mathbf{E}^Y -related to $\iota_{X,Y}(a')$. If instead $a' = a_{\bar{j}}$ for some $\bar{j} \in F_u$, then $\iota_{X,Y}(a') = b_{\bar{j}}$, and hence

$$a = a_{\bar{i}} \mathbf{E}^X a_{\bar{j}} = a' \Leftrightarrow \bar{i} \mathbf{E}^{F_u} \bar{j} \Leftrightarrow \iota_{X,Y}(a) = b_{\bar{i}} \mathbf{E}^Y b_{\bar{j}} = \iota_{X,Y}(a'). \quad \square$$

Recall from (12.5) that for every $X \in \text{Mod}_{\Psi}^{\kappa}$ and $(u, \alpha) \in {}^{<\omega}2 \times \kappa$, we have set $g(X)(u, \alpha) = 1$ if and only if X satisfies the $\mathcal{L}_{\kappa+\kappa}$ -sentence

$$\exists \langle x_i \mid i \in F_u \rangle [F_u(\langle x_i \mid i \in F_u \rangle) \wedge \text{Seq}_s(x_\emptyset)]$$

where $s = s_\alpha \in {}^{<\omega}\kappa$ is such that $\langle\langle s \rangle\rangle = \alpha$. For technical reasons we need to replace such a sentence with one belonging to the bounded version $\mathcal{L}_{\kappa+\kappa}^b$ of $\mathcal{L}_{\kappa+\kappa}$ (see Definition 8.3(ii)) — this will be crucial for the results in Section 14.1.

LEMMA 12.8. *Let $X \in \text{Mod}_{\Psi}^{\kappa}$ and $(u, \alpha) \in {}^{<\omega}2 \times \kappa$. Then $g(X)(u, \alpha) = 1 \Leftrightarrow X \models \sigma_{u, \alpha}$, where $\sigma_{u, \alpha}$ is the $\mathcal{L}_{\kappa+\kappa}^b$ -sentence*

$$\begin{aligned} (\sigma_{u, \alpha}) \quad & \exists x \exists y \left[\text{Seq}(x) \wedge \widehat{\text{Seq}}(y) \wedge x \mathbf{E} y \wedge \exists \langle z_i \mid i \in U_{\alpha} \rangle \left(y \mathbf{E} z_{\emptyset} \wedge \tau_{\text{qf}}(U_{\alpha})(\langle z_i \mid i \in U_{\alpha} \rangle) \right) \right. \\ & \wedge \neg \exists \langle w_j \mid j \in U_{\alpha+1} \rangle \left(y \mathbf{E} w_{\emptyset} \wedge \tau_{\text{qf}}(U_{\alpha+1})(\langle w_j \mid j \in U_{\alpha+1} \rangle) \right) \\ & \left. \wedge \exists \langle l_k \mid k \in F_u^{\theta(u)} \rangle \left(l_{\emptyset} \simeq x \wedge \left(\bigwedge_{\emptyset \neq k \in F_u^{\theta(u)}} F(l_k) \right) \wedge \tau_{\text{qf}}(F_u^{\theta(u)})(\langle l_k \mid k \in F_u^{\theta(u)} \rangle) \right) \right], \end{aligned}$$

where we set $F_u^{\theta(u)} := F_u \cap {}^{\theta(u)+1}2$.

PROOF. Using Proposition 10.5(c), it is not hard to see that if $g(X)(u, \alpha) = 1$ then $X \models \sigma_{u, \alpha}$. For the other direction, assume that $X \models \sigma_{u, \alpha}$, and let $a, \hat{a}, \langle b_i \mid i \in U_{\alpha} \rangle$, and $\langle c_k \mid k \in F_u^{\theta(u)} \rangle$ be (sequences of) elements of X such that

$$\begin{aligned} X \models & \text{Seq}[a] \wedge \widehat{\text{Seq}}[\hat{a}] \wedge \left(a \mathbf{E} \hat{a} \wedge \hat{a} \mathbf{E} b_{\emptyset} \wedge \tau_{\text{qf}}(U_{\alpha})(\langle b_i \mid i \in U_{\alpha} \rangle) \right) \\ & \wedge \neg \exists \langle w_j \mid j \in U_{\alpha+1} \rangle \left(\hat{a} \mathbf{E} w_{\emptyset} \wedge \tau_{\text{qf}}(U_{\alpha+1})(\langle w_j \mid j \in U_{\alpha+1} \rangle) \right) \\ & \wedge \left(c_{\emptyset} \simeq a \wedge \left(\bigwedge_{\emptyset \neq k \in F_u^{\theta(u)}} F[c_k] \right) \wedge \tau_{\text{qf}}(F_u^{\theta(u)})(\langle c_k \mid k \in F_u^{\theta(u)} \rangle) \right). \end{aligned}$$

It follows from $X \models \Phi_2 \wedge \Phi_3$ and $X \models \text{Seq}[a] \wedge \widehat{\text{Seq}}[\hat{a}] \wedge a \mathbf{E} \hat{a}$ that $X \models \text{Seq}_s[a]$ and $X \models \widehat{\text{Seq}}_s[\hat{a}]$ for one and the same $s \in {}^{<\omega}\kappa$: we claim that $\langle\langle s \rangle\rangle = \alpha$. Indeed, from $X \models \tau_{\text{qf}}(U_{\alpha})(\langle b_i \mid i \in U_{\alpha} \rangle)$ and $\hat{a} \mathbf{E}^X b_{\emptyset}$ it easily follows that $X \models U[b_i] \wedge \text{root}[b_i, \hat{a}]$ for every $i \in U_{\alpha}$. Arguing by contradiction, one sees that Lemma 10.5(c) together with $X \models \widehat{\text{Seq}}_s[\hat{a}]$ and $X \models \tau_{\text{qf}}(U_{\alpha})(\langle b_i \mid i \in U_{\alpha} \rangle)$ imply $\langle\langle s \rangle\rangle \geq \alpha$. Moreover, by Lemma 10.5(b) we get that $\langle\langle s \rangle\rangle > \alpha$ would contradict

$$X \models \neg \exists \langle w_j \mid j \in U_{\alpha+1} \rangle \left(\hat{a} \mathbf{E} w_{\emptyset} \wedge \tau_{\text{qf}}(U_{\alpha+1})(\langle w_j \mid j \in U_{\alpha+1} \rangle) \right).$$

Therefore, $\langle\langle s \rangle\rangle = \alpha$. Since $X \models c_{\emptyset} \simeq a$, we have also showed that $X \models \text{Seq}_s[c_{\emptyset}]$ for the unique $s \in {}^{<\omega}\kappa$ such that $\langle\langle s \rangle\rangle = \alpha$.

Fix any $\emptyset \neq k \in F_u^{\theta(u)}$. Since $X \models F[c_k]$, by $X \models \Phi_5$ we get that there are $v_k \in {}^{<\omega}2$ and a sequence $\langle d_i^k \mid i \in F_{v_k} \rangle$ of elements of X such that $X \models F_{v_k}[\langle d_i^k \mid i \in F_{v_k} \rangle]$ and $X \models c_k \simeq d_{i_k}^k$ for some $\emptyset \neq i_k \in F_{v_k}$. In particular, $X \models \text{Seq}[d_{\emptyset}^k]$: we claim that $d_{\emptyset}^k = c_{\emptyset}$. If not, then by $X \models \bigwedge_{i \leq 4} \Phi_i$ and Lemma 12.5 there would be a path of length ≥ 2 connecting c_{\emptyset} to d_{\emptyset}^k which is totally contained in $\mathbb{G}_0(X) \subseteq \mathbb{G}_1(X)$. On the other hand, the set of vertices $\{c_{k'} \mid k' \subseteq k\} \cup \{d_i^k \mid i \subseteq i_k\}$ would contain a path of length ≥ 2 connecting c_{\emptyset} to d_{\emptyset}^k which, except for its extreme points, is totally contained in $F(X)$. Since $\mathbb{G}_1(X) \cap F(X) = \emptyset$ by Remark 12.4, the two paths would then be distinct, contradicting the fact that X is acyclic by $X \models \Phi_0$. A similar argument shows that by acyclicity of X the two paths $\langle c_{k'} \mid k' \subseteq k \rangle$ and $\langle d_i^k \mid i \subseteq i_k \rangle$ joining $c_{\emptyset} = d_{\emptyset}^k$ to $c_k = d_{i_k}^k$ must coincide. Thus, in particular, $c_{\langle \emptyset \rangle} = d_{\langle \emptyset \rangle}^k$ for every $\emptyset \neq k \in F_u^{\theta(u)}$. Fix now $k, k' \in F_u^{\theta(u)} \setminus \{\emptyset\}$. Since $d_{\langle \emptyset \rangle}^k = c_{\langle \emptyset \rangle} = d_{\langle \emptyset \rangle}^{k'}$, by $X \models \Phi_7$ it follows that

$$\{d_i^k \mid i \in F_{v_k}\} = \{d_i^{k'} \mid i \in F_{v_{k'}}\}.$$

Since the maps $i \mapsto d_i^k$ and $i \mapsto d_i^{k'}$ witness that both F_{v_k} and $F_{v_{k'}}$ are isomorphic to the same structure $X \upharpoonright \{d_i^k \mid i \in F_{v_k}\} = X \upharpoonright \{d_i^{k'} \mid i \in F_{v_{k'}}\}$, we have $F_{v_k} \cong F_{v_{k'}}$, whence $v_k = v_{k'}$ by (10.5c). It then follows from $X \models \Phi_8$ and $d_{\langle 0 \rangle}^k = d_{\langle 0 \rangle}^{k'}$ that $d_i^k = d_i^{k'}$ for all $i \in F_{v_k} = F_{v_{k'}}$. Set $v := v_k$ and $d_i := d_i^k$ for some/any $\emptyset \neq k \in F_u^{\theta(u)}$ and all $i \in F_v$. Since $\{c_k \mid k \in F_u^{\theta(u)}\} \subseteq \{d_i \mid i \in F_v\}$ and $c_{0^{\theta(u)}}$ has three distinct neighbors among the c_k 's, necessarily $c_{0^{\theta(u)}} = d_{0^{\theta(v)}}$. From this and $c_\emptyset = d_\emptyset$ it also follows that $\{c_k \mid k \subseteq 0^{\theta(u)}\} = \{d_i \mid i \subseteq 0^{\theta(v)}\}$ (here we use again the fact that X is acyclic), whence $\theta(u) = \theta(v)$, and hence also $u = v$ by injectivity of θ . Since $d_\emptyset = c_\emptyset$ and $X \models \text{Seq}_s[c_\emptyset]$, it thus follows that $\langle d_i \mid i \in F_u \rangle$ witnesses that

$$X \models \exists \langle x_i \mid i \in F_u \rangle [\mathbf{F}_u(\langle x_i \mid i \in F_u \rangle) \wedge \text{Seq}_s(x_\emptyset)],$$

and since $\langle\langle s \rangle\rangle = \alpha$ we have $g(X)(u, \alpha) = 1$, as desired. \square

12.3. The invariant universality of \sqsubset_{CT}^κ . Endow ${}^{<\omega}2^{\times \kappa}2$ with the *product* topology τ_p , so that $({}^{<\omega}2^{\times \kappa}2, \tau_p)$ is homeomorphic to $({}^\kappa 2, \tau_p)$ (see Section 7.2.1 and, in particular, Example 7.8(A)).

LEMMA 12.9. *For every (τ_p) -open set $V \subseteq {}^{<\omega}2^{\times \kappa}2$ there is an $\mathcal{L}_{\kappa+\kappa}^b$ -sentence σ_V such that $g^{-1}(V) = \text{Mod}_{\sigma_V}^\kappa \cap \text{Mod}_\Psi^\kappa$ (equivalently, $g^{-1}(V) = \text{Mod}_{\sigma_V \wedge \Psi}^\kappa$).*

The fact that the sentence σ_V belongs to the fragment $\mathcal{L}_{\kappa+\kappa}^b$ (and not just to $\mathcal{L}_{\kappa+\kappa}$) will be crucially used in Section 14.1 to prove that certain maps are (effective) $\kappa + 1$ -Borel — see Lemma 14.7 and Theorem 14.8.

PROOF. Recall from (7.1) that $\mathcal{S} = \{\tilde{N}_{(u,\alpha),i}^A \mid (u,\alpha) \in {}^{<\omega}2^{\times \kappa} \wedge i = 0, 1\}$ is a subbasis for the product topology on ${}^A 2$, where $A := {}^{<\omega}2^{\times \kappa}$. The map $V \mapsto \sigma_V$ will be defined first on \mathcal{S} , then on the canonical basis \mathcal{B}_p (which is generated by \mathcal{S} by taking finite intersections), and finally on all τ_p -open sets.

When $V = \tilde{N}_{(u,\alpha),i}^A$ let σ_V be $\sigma_{u,\alpha}$ if $i = 1$ or $\neg \sigma_{u,\alpha}$ otherwise. By Lemma 12.8 we get $g^{-1}(V) = \text{Mod}_{\sigma_V}^\kappa \cap \text{Mod}_\Psi^\kappa$.

Pick $V \in \mathcal{B}_p$ and let $U_1, \dots, U_n \in \mathcal{S}$ be such that $V = U_1 \cap \dots \cap U_n$. Then $g^{-1}(V) = \text{Mod}_{\sigma_V}^\kappa \cap \text{Mod}_\Psi^\kappa$, where σ_V is $\sigma_{U_1} \wedge \dots \wedge \sigma_{U_n}$.

Given now an arbitrary τ_p -open set $V \subseteq {}^{<\omega}2^{\times \kappa}2$, let B_V be the collection of those $U \in \mathcal{B}_p$ which are contained in V : since the cardinality of B_V is at most $|\kappa|^{<\omega} = \kappa$ (which is the cardinality of \mathcal{B}_p), we get the desired result letting σ_V be $\bigvee_{U \in B_V} \sigma_U$. \square

LEMMA 12.10. *Let $T \in \mathbb{T}_\kappa$ and f_T be the map from (11.5). Then the map $g \circ f_T: {}^\omega 2 \rightarrow {}^{<\omega}2^{\times \kappa}2$ is continuous.*

PROOF. Fix $(u, \alpha) \in A = {}^{<\omega}2^{\times \kappa}$, and let $s \in {}^{<\omega} \kappa$ be such that $\langle\langle s \rangle\rangle = \alpha$. Let

$$D_1 := \left\{ v \in {}^{\text{lh } u} 2 \mid (u, v, s) \in \tilde{T} \right\},$$

where \tilde{T} is the tree obtained from T as in (11.3), and let $D_0 := {}^{\text{lh } u} 2 \setminus D_1$. Then by definition of f_T and g we get $(g \circ f_T)^{-1}(\tilde{N}_{(u,\alpha),i}^A) = \bigcup_{v \in D_i} N_v$. \square

COROLLARY 12.11. *Let $T \in \mathbb{T}_\kappa$ and f_T be as in (11.5). For every closed set $C \subseteq {}^\omega 2$, $(g \circ f_T)(C)$ is closed.*

PROOF. The set C is compact since it is a closed subset of the compact space ${}^\omega 2$. Therefore, since $g \circ f_T$ is continuous by Lemma 12.10, $(g \circ f_T)(C)$ is compact as well: but then it is also closed because ${}^{<\omega 2 \times \kappa} 2$ is a Hausdorff space. \square

COROLLARY 12.12. *Let $T \in \mathbb{T}_\kappa$ and f_T be as in (11.5). Then there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ_T such that the closure under isomorphism of $\text{ran}(f_T)$ is $\text{Mod}_{\sigma_T}^\kappa$, i.e.*

$$\text{Mod}_{\sigma_T}^\kappa = \{X \in \text{Mod}_{\mathcal{L}}^\kappa \mid X \cong f_T(x) \text{ for some } x \in {}^\omega 2\}.$$

Notice that since $\text{ran}(f_T) \subseteq \text{Mod}_{\Psi}^\kappa$ by Lemma 12.6, we thus have in particular (see Figure 5)

$$\text{ran}(f_T) \subseteq \text{Mod}_{\sigma_T}^\kappa \subseteq \text{Mod}_{\Psi}^\kappa.$$

PROOF. Let V be the complement of $\text{ran}(g \circ f_T) = (g \circ f_T)({}^\omega 2)$, which is open by Corollary 12.11. Use Lemma 12.9 to find an $\mathcal{L}_{\kappa+\kappa}^b$ -sentence $\hat{\sigma} := \sigma_V$ such that $g^{-1}(V) = \text{Mod}_{\hat{\sigma}}^\kappa \cap \text{Mod}_{\Psi}^\kappa$. Finally, let σ_T be $\Psi \wedge \neg \hat{\sigma}$, so that

$$(12.8) \quad g^{-1}(\text{ran}(g \circ f_T)) = \text{Mod}_{\sigma_T}^\kappa.$$

If $X \in \text{Mod}_{\mathcal{L}}^\kappa$ is such that $X \cong f_T(x)$ for some $x \in {}^\omega 2$, then $X \in \text{Mod}_{\Psi}^\kappa$ (use the fact that $f_T(X) \in \text{Mod}_{\Psi}^\kappa$ by Lemma 12.6 and that Mod_{Ψ}^κ is invariant under isomorphism): therefore $g(X)$ is defined and equals $(g \circ f_T)(x)$ by Proposition 12.7, so that $X \in g^{-1}(\text{ran}(g \circ f_T))$ and hence $X \models \sigma_T$ by (12.8). Conversely, if $X \in \text{Mod}_{\sigma_T}^\kappa = \text{Mod}_{\Psi \wedge \neg \hat{\sigma}}^\kappa \subseteq \text{Mod}_{\Psi}^\kappa$ then by (12.8) there must be $x \in {}^\omega 2$ such that $g(X) = (g \circ f_T)(x)$: but then $X \cong f_T(x)$ by Proposition 12.7 again. \square

We now define the last reduction $h_T: \text{Mod}_{\sigma_T}^\kappa \rightarrow {}^\omega 2$ of Figure 5.

DEFINITION 12.13. Let $T \in \mathbb{T}_\kappa$ be such that $R = p[T]$ is a quasi-order, f_T be as in (11.5), and σ_T as in Corollary 12.12. Then we define the map $h_T: \text{Mod}_{\sigma_T}^\kappa \rightarrow {}^\omega 2$ by setting for all $X \in \text{Mod}_{\sigma_T}^\kappa$

$$(12.9) \quad h_T(X) = x \Leftrightarrow f_T(x) \cong X.$$

Notice that the function h_T is well-defined: by Corollary 12.12, if $X \models \sigma_T$ then there is at least one x satisfying (12.9), and by Theorem 11.8(b) such x is unique because if $y \in {}^\omega 2$ is distinct from x , then $f_T(x) \not\cong f_T(y)$, whence $f_T(y) \not\cong X$.

COROLLARY 12.14. *Let $T \in \mathbb{T}_\kappa$, f_T be as in (11.5), and σ_T be as in Corollary 12.12. If $R = p[T]$ is a quasi-order, then the map $h_T: \text{Mod}_{\sigma_T}^\kappa \rightarrow {}^\omega 2$ from Definition 12.13 simultaneously reduces \sqsubset to R and \cong to $=$.*

PROOF. Let $X, Y \in \text{Mod}_{\sigma_T}^\kappa$ and assume that $R = p[T]$ is a quasi-order. Let $x := h_T(X)$ and $y := h_T(Y)$. Since $f_T(x) \cong X$ and $f_T(y) \cong Y$ by (12.9), by Theorem 11.8 we have

$$X \sqsubset Y \Leftrightarrow f_T(x) \sqsubset f_T(y) \Leftrightarrow x R y,$$

and

$$X \cong Y \Leftrightarrow f_T(x) \cong f_T(y) \Leftrightarrow x = y. \quad \square$$

Summing up what we obtained in this section, we have the following theorem (see also Figure 5).

THEOREM 12.15. *Let κ be an infinite cardinal and $T \in \mathbb{T}_\kappa$. If $R = p[T]$ is a quasi-order, then there are σ_T, f_T, h_T such that:*

- (a) σ_T is an $\mathcal{L}_{\kappa+\kappa}$ sentence;
- (b) f_T reduces R to $\sqsubset_{\sigma_T}^\kappa$ and $=$ to $\cong_{\sigma_T}^\kappa$;
- (c) h_T reduces $\sqsubset_{\sigma_T}^\kappa$ to R and $\cong_{\sigma_T}^\kappa$ to $=$;
- (d) $h_T \circ f_T = \text{id}$ and $(f_T \circ h_T)(X) \cong X$ for every $X \in \text{Mod}_{\sigma_T}^\kappa$.

In particular, $\sqsubset_{\text{CT}}^\kappa$ is invariantly universal for κ -Souslin quasi-orders on ${}^\omega 2$, that is to say: for every κ -Souslin quasi-order R on ${}^\omega 2$ there is an $\mathcal{L}_{\kappa^+ \kappa}$ -sentence σ all of whose models are combinatorial trees such that $R \sim \sqsubset_\sigma^\kappa$.

PROOF. For part (d), observe that for every $x \in {}^\omega 2$ we must have $f_T((h_T \circ f_T)(x)) \cong f_T(x)$ by (12.9), whence $(h_T \circ f_T)(x) = x$ by part (b). The other condition on $f_T \circ h_T$ is just a rewriting of (12.9), so we are done. \square

Since $R \sim \sqsubset_{\sigma_T}^\kappa$ by (b) and (c), it follows that the quotient order of R is bi-embeddable with the quotient order of $\sqsubset_{\sigma_T}^\kappa$. However, (d) implies the following stronger result.

COROLLARY 12.16. *Let κ be an infinite cardinal and $T \in \mathbb{T}_\kappa$. If $R = p[T]$ is a quasi-order, then there is an $\mathcal{L}_{\kappa^+ \kappa}$ -sentence σ_T (all of whose models are combinatorial trees) such that the quotient orders of R and $\sqsubset_{\sigma_T}^\kappa$ are isomorphic.*

Notice also that using Corollary 12.16 all results on invariant universality of Sections 14, 15, and 16.2.1 could be reformulated in terms of (definable) isomorphisms between the induced quotient orders.

REMARKS 12.17. (i) It is not hard to check that all models of the $\mathcal{L}_{\kappa^+ \kappa}$ -sentence Ψ , and hence also the models of the sentence σ_T from Corollary 12.12, are necessarily of size κ . Therefore Ψ characterizes the cardinal κ in the sense of [Kni77, p. 59], once in the original definition we replace the logic $\mathcal{L}_{\omega_1 \omega}$ with the more powerful $\mathcal{L}_{\kappa^+ \kappa}$. This also implies that $\text{Mod}_{\sigma_T}^\infty = \text{Mod}_{\sigma_T}^\kappa$, so that the conclusion of Theorem 12.15 could also be reformulated as follows:

For every κ -Souslin quasi-order R on ${}^\omega 2$ there is an $\mathcal{L}_{\kappa^+ \kappa}$ -sentence σ all of whose models are combinatorial trees such that $R \sim \sqsubset_\sigma^\infty$.

Notice that all results on invariant universality of Sections 14, 15, and 16.2.1 could be reformulated analogously.

- (ii) Every model X of Ψ (and hence every model of σ_T for $T \in \mathbb{T}_\kappa$) admits an $\mathcal{L}_{\kappa^+ \kappa}$ -**Scott sentence**, i.e. an $\mathcal{L}_{\kappa^+ \kappa}$ -sentence σ^X such that for every $Y \in \text{Mod}_{\mathcal{L}}^\kappa$

$$Y \models \sigma^X \Leftrightarrow Y \cong X.$$

The sentence σ^X is obtained by applying an argument similar to that of Corollary 12.12: Let $V := {}^{<\omega} 2^{\times \kappa} 2 \setminus \{g(X)\}$ (which is open since ${}^{<\omega} 2^{\times \kappa} 2$ is a Hausdorff space), and let σ^X be the $\mathcal{L}_{\kappa^+ \kappa}$ -sentence $\Psi \wedge \neg \sigma_V$, where σ_V is as in Lemma 12.9.

It is a well-known classical result due to D. Scott that when $\kappa = \omega$ then Remark 12.17(ii) is true for every $X \in \text{Mod}_{\mathcal{L}}^\omega$, see e.g. [Kec95, Corollary 16.10]. However, when κ is uncountable such condition may fail for some $X \in \text{Mod}_{\mathcal{L}}^\kappa$, as the next example, due to S. D. Friedman, shows.³⁴

EXAMPLE 12.18. Let \mathcal{L} be the order language consisting of one binary relational symbol, set $\kappa = \omega_1$, and consider the collection of all ω_1 -like dense linear orders on ω_1 without a minimum (briefly, ω_1 -DLO), where a linear order is called ω_1 -like if all its proper initial segments are countable. We will show that there are two non-isomorphic ω_1 -DLO X, Y which are $\mathcal{L}_{\omega_2 \omega_1}$ -equivalent, i.e. they satisfy the same $\mathcal{L}_{\omega_2 \omega_1}$ -sentences: this implies that none of X and Y admits an $\mathcal{L}_{\omega_2 \omega_1}$ -Scott sentence. Equip $\mathbb{Q}_0 := \mathbb{Q} \cap (0; 1)$, $\mathbb{Q}_1 := \mathbb{Q} \cap [0; 1)$, and ω_1 with the usual orders, and let

$$X := \omega_1 \times \mathbb{Q}_0$$

³⁴Working in ZFC and assuming $\kappa^{<\kappa} = \kappa > \omega$, easier counterexamples may be isolated. However, the construction in Example 12.18 has the further merit of being carried out in ZF alone and without further assumptions on $\kappa = \omega_1$ (in particular, Example 12.18 works also in models of ZFC + $\neg \text{CH}$).

and

$$Y := \{0\} \times \mathbb{Q}_0 \cup (\omega_1 \setminus \{0\}) \times \mathbb{Q}_1$$

be endowed with the lexicographical ordering. Then, after identifying them with corresponding structures on ω_1 , X and Y are easily seen to be ω_1 -DLO. Moreover, they are not isomorphic because Y contains a closed unbounded (with respect to its ordering) set of order type ω_1 , while X does not contain such a substructure. To show that X and Y are $\mathcal{L}_{\omega_2\omega_1}$ -equivalent, recall from e.g. [FHK14, Theorem 10 and Remark 11] that two structures X and Y of size ω_1 are $\mathcal{L}_{\omega_2\omega_1}$ -equivalent if and only if Player **II** has a winning strategy in the following Ehrenfeuch-Fraïssé game $\text{EF}_{\omega}^{\omega_1}(X, Y)$ (see also [Vää11] for more on these games). Set $A_{-1} = B_{-1} := \emptyset$, and let f_{-1} be the empty function. At the n -th turn, Player **I** picks countable sets A_n, B_n with $A_{n-1} \subseteq A_n \subseteq X$ and $B_{n-1} \subseteq B_n \subseteq Y$, while Player **II** plays a partial function f_n between X and Y extending f_{n-1} whose domain and range contain, respectively, A_n and B_n . After ω -many turns, we say that Player **II** wins if and only if $f := \bigcup_{n \in \omega} f_n$ is a partial isomorphism between X and Y . It is not hard to check that if X and Y are the ω_1 -DLO defined above then **II** has a winning strategy in $\text{EF}_{\omega}^{\omega_1}(X, Y)$. Indeed, **II** can always assume without loss of generality that at the n -th turn **I** is choosing infinite initial segments A_n and B_n of, respectively, X and Y with the further property that $A_n \setminus A_{n-1}$ and $B_n \setminus B_{n-1}$ are infinite initial segments without maximum and minimum of, respectively, $X \setminus A_{n-1}$ and $Y \setminus B_{n-1}$. Under such assumption, **II** can then extend f_{n-1} to f_n by choosing any isomorphism between $A_n \setminus A_{n-1}$ and $B_n \setminus B_{n-1}$, which exists because both sets are isomorphic to the rationals $\mathbb{Q} = (\mathbb{Q}, \leq)$.

12.4. More absoluteness results. Continuing the work in Section 11.4 on absoluteness of the definition of the map f_T from (11.5), we are now going to observe that also the $\mathcal{L}_{\kappa+\kappa}$ -sentence σ_T from Corollary 12.12 and the map h_T from Definition 12.13 are (essentially) absolute between transitive models of **ZF** containing all the relevant parameters. These results, together with those from Section 11.4, will be crucially used in Section 14.2.

Given an arbitrary transitive model M of **ZF**, $\kappa \in \text{Card}^M$, and $T \in (\mathbb{T}_{\kappa})^M$, carry out the constructions from Section 12 inside M (this is possible by our choice of M), i.e. let

$$g^M := (g)^M : (\text{Mod}_{\Psi}^{\kappa})^M \rightarrow ({}^{<\omega}2^{\times \kappa}2)^M$$

be defined as in (12.4)–(12.5), and let

$$\sigma_T^M := (\sigma_T)^M \in (\mathcal{L}_{\kappa+\kappa})^M$$

be the sentence obtained in the proof of Corollary 12.12 (where all objects involved are now computed in M). Note that by Corollary 12.12, which holds in M , we have that $(\text{Mod}_{\sigma_M}^{\kappa})^M$ is the saturation of the range of the map f_T^M from Section 11.4 (as computed in M), and that $M \models \text{“Mod}_{\sigma_M}^{\kappa} = (g^M)^{-1}(\text{ran}(g^M \circ f_T^M))\text{”}$. With this notation, we then get the following absoluteness result for the sentence σ_T .

PROPOSITION 12.19. *Let M_0, M_1 be transitive models of **ZF**, and let κ and T be such that $\kappa \in \text{Card}^{M_i}$ and $T \in (\mathbb{T}_{\kappa})^{M_i}$ (for $i = 0, 1$). Then $\sigma_T^{M_0} = \sigma_T^{M_1}$.*

PROOF. First notice that the set $\mathbf{Fn}(<^{\omega}2 \times \kappa, 2; \omega)$ is absolute for transitive models of **ZF** containing κ . Since the elements of this set determine (both in M_0 and in M_1) the canonical basis $\mathcal{B}_p = \mathcal{B}_p(<^{\omega}2 \times \kappa, 2)$ for the product topology τ_p on $<^{\omega}2 \times \kappa, 2$, by the way we defined $\sigma_T^{M_i}$ it is enough to show that for every $s \in \mathbf{Fn}(<^{\omega}2 \times \kappa, 2; \omega)$

$$(12.10) \quad M_0 \models \text{“ran}(g^{M_0} \circ f_T^{M_0}) \cap \mathbf{N}_s^A \neq \emptyset\text{”} \quad \text{if and only if} \quad M_1 \models \text{“ran}(g^{M_1} \circ f_T^{M_1}) \cap \mathbf{N}_s^A \neq \emptyset\text{”},$$

where $A := <^{\omega}2 \times \kappa$ (see Section 7.2.1). In fact, in the proof of Corollary 12.12 we have set $\sigma_T^{M_i} := \Psi \wedge \neg \sigma_{V_i}^{M_i}$ with $V_i = ({}^A 2)^{M_i} \setminus \text{ran}(g^{M_i} \circ f_T^{M_i})$, where $\sigma_{V_i}^{M_i}$ is computed in M_i according

to the proof of Lemma 12.9. By inspecting such proof, it is easy to check that the $(\mathcal{L}_{\kappa+\kappa})^{M_i}$ -sentence $\sigma_{V_i}^{M_i}$ only depends on the set of $s \in \mathbf{Fn}(<\omega 2 \times \kappa, 2; \omega)$ for which $M_i \models \mathbf{N}_s^A \subseteq V_i$: thus if the equivalence in (12.10) is true, this set is the same when computed in M_0 or M_1 , whence

$$\sigma_T^{M_0} = \Psi \wedge \neg \sigma_{V_0}^{M_0} = \Psi \wedge \neg \sigma_{V_1}^{M_1} = \sigma_T^{M_1},$$

as desired.

We now prove (12.10), starting with the implication from left to right. By Lemma 12.10 (which holds in M_0), the function $g^{M_0} \circ f_T^{M_0}$ is continuous in M_0 . Therefore, if $M_0 \models \text{“ran}(g^{M_0} \circ f_T^{M_0}) \cap \mathbf{N}_s^A \neq \emptyset\text{”}$, then there is $t \in <\omega 2$ such that $M_0 \models \text{“}\mathbf{N}_t^\omega \subseteq (g^{M_0} \circ f_T^{M_0})^{-1}(\mathbf{N}_s^A)\text{”}$, in particular $M_0 \models \text{“}(g^{M_0} \circ f_T^{M_0})(\bar{x}) \in \mathbf{N}_s^A\text{”}$ with $\bar{x} := t \cap 0^{(\omega)}$. Since $(<\omega 2)^{M_0} = (<\omega 2)^{M_1}$, it follows that $\bar{x} \in (\omega 2)^{M_0} \cap (\omega 2)^{M_1}$ and hence $f_T^{M_0}(\bar{x}) = f_T^{M_1}(\bar{x})$ by Proposition 11.10(b). By Lemma 12.6 (which holds in both M_0 and M_1) and the definition of the map g given in (12.4)–(12.5), both g^{M_0} and g^{M_1} are defined on $X := f_T^{M_0}(\bar{x}) = f_T^{M_1}(\bar{x})$ and $g^{M_0}(X) = g^{M_1}(X)$. Therefore $M_1 \models \text{“}(g^{M_1} \circ f_T^{M_1})(\bar{x}) \in \mathbf{N}_s^A\text{”}$ as well, witnessing $M_1 \models \text{“ran}(g^{M_1} \circ f_T^{M_1}) \cap \mathbf{N}_s^A \neq \emptyset\text{”}$.

The reverse implication is proved in a similar way by switching the role of M_0 and M_1 . \square

Let M , κ , T , and σ_T^M be as in the paragraph preceding Proposition 12.19. Applying Definition 12.13 in M , we also get the map

$$h_T^M := (h_T)^M : (\text{Mod}_{\sigma_T^M}^\kappa)^M \rightarrow (\omega 2)^M.$$

Notice that since Corollary 12.14 holds in M , the map h_T^M reduces, in the sense of M , the embeddability relation $(\sqsubset_{\sigma^M}^\kappa)^M$ to $R^M := (\text{p}[T])^M$ (as long as R^M is a quasi-order in M).

PROPOSITION 12.20. (a) *There is an LST-formula $\Psi_{h_T}(x_0, x_1, z_0, z_1)$ such that for every transitive model M of ZF with $\kappa \in \text{Card}^M$ and $T \in (\mathbb{T}_\kappa)^M$, the graph of h_T^M is defined in M by $\Psi_{h_T}(x_0, x_1, \kappa, T)$, that is: for every $X \in (\text{Mod}_{\sigma_T^M}^\kappa)^M$ and $x \in (\omega 2)^M$*

$$h_T^M(X) = x \Leftrightarrow M \models \Psi_{h_T}[X, x, \kappa, T].$$

(b) *Let M_0, M_1 be transitive models of ZF, and let κ and T be such that $\kappa \in \text{Card}^{M_i}$ and $T \in (\mathbb{T}_\kappa)^{M_i}$ (for $i = 0, 1$), so that $\sigma_T^{M_0} = \sigma_T^{M_1}$ by Proposition 12.19. If $(\kappa \kappa)^{M_0} \subseteq (\kappa \kappa)^{M_1}$, then setting $\sigma := \sigma_T^{M_0} = \sigma_T^{M_1}$ we have that for every $X \in (\text{Mod}_\sigma^\kappa)^{M_0} \cap (\text{Mod}_\sigma^\kappa)^{M_1}$*

$$(12.11) \quad h_T^{M_0}(X) = h_T^{M_1}(X).$$

PROOF. For part (a) it is enough to observe that equation (12.9) is rendered by the LST-formula $\Psi_{h_T}(x_0, x_1, z_0, z_1)$

$$x_0 \in \text{Mod}_{\mathcal{L}}^{z_0} \wedge x_1 \in \omega 2 \wedge \exists i \in \text{Sym}(z_0) \forall x_2 (\Psi_{f_T}(x_1, x_2, z_0, z_1) \Rightarrow i \text{ witnesses } x_2 \cong x_0),$$

where $\Psi_{f_T}(x_1, x_2, z_0, z_1)$ is as in Proposition 11.10 and “ i witnesses $x_2 \cong x_0$ ” stands for

$$(12.12) \quad \forall \alpha, \beta < \kappa (\alpha \mathbf{E}^{x_2} \beta \Leftrightarrow i(\alpha) \mathbf{E}^{x_0} i(\beta)).$$

We now prove part (b). Let $X \in (\text{Mod}_\sigma^\kappa)^{M_0} \cap (\text{Mod}_\sigma^\kappa)^{M_1}$ and set $x := h_T^{M_1}(X) \in (\omega 2)^{M_1}$ and $x' := h_T^{M_0}(X) \in (\omega 2)^{M_0} \subseteq (\omega 2)^{M_1}$ (the latter inclusion follows from our assumption $(\kappa \kappa)^{M_0} \subseteq (\kappa \kappa)^{M_1}$). Then there is $i \in (\text{Sym}(\kappa))^{M_0}$ such that $M_0 \models \text{“}i \text{ witnesses } f_T^{M_0}(x') \cong X\text{”}$. Notice that since $(\text{Sym}(\kappa))^{M_0} \subseteq (\text{Sym}(\kappa))^{M_1}$ (again by our assumption $(\kappa \kappa)^{M_0} \subseteq (\kappa \kappa)^{M_1}$), then $i \in (\text{Sym}(\kappa))^{M_1}$ as well. Since $f_T^{M_0}(x') = f_T^{M_1}(x')$ by Proposition 11.10(b), and since $f_T^{M_1}(x') \in M_1$, the sentence “ i is an isomorphism between the structures $f_T^{M_0}(x')$ and $X \in \text{Mod}_{\mathcal{L}}^\kappa$ ” can be formalized by the Δ_0 LST-formula with parameters $i, f_T^{M_0}(x'), X$, and κ in $M_0 \cap M_1$ provided in (12.12) (where we replace x_2 and x_0 by, respectively, $f_T^{M_0}(x')$ and X), and is therefore absolute between M_0 and M_1 . Therefore $M_1 \models \text{“}f_T^{M_1}(x') \cong X\text{”}$. By our choice of x we also have

$M_1 \models "f_T^{M_1}(x) \cong X"$, and therefore it follows from Theorem 11.8(b), which holds in the ZF-model M_1 , that $x' = x$. Thus (12.11) is satisfied and we are done. \square

REMARK 12.21. Remark 11.11 applies also to the results of this section: to simplify the presentation, Propositions 12.19 and 12.20 were again formulated as second-order statements (as the transitive ZF-models M , M_0 , and M_1 they involve may be proper classes), but they can be reformulated as purely first-order statements by restricting the kind of models on which M , M_0 , and M_1 can vary — see the discussion in Section 14.2, and in particular Examples 14.17.

13. An alternative approach

As explained at the beginning of Section 10, two different approaches have been employed in the literature to show that the embeddability relation on countable structures is invariantly universal, namely the approaches (1) and (2) briefly described on page 72. In Sections 10–12 we successfully followed approach (2), and this will provide generalizations to uncountable cardinals κ of Theorems 1.1 and 1.4 (see Sections 14 and 15). However, our proof of Theorem 12.15 does not yield a generalization of Theorem 1.5: the reason is that the $\mathcal{L}_{\kappa+\kappa}$ -sentence σ_T from Corollary 12.12 is quite complicated, and hence $\text{Mod}_{\sigma_T}^\kappa$ is usually far from being a $\kappa + 1$ -Borel subset of $\text{Mod}_{\mathcal{L}}^\kappa$ (unless we assume $\text{AC} + \kappa^{<\kappa} = \kappa$, see Section 8.2 and, in particular, Remark 8.12).

In this section we are going to show that although approach (1) forces us to consider a less natural kind of structures, it allows us to further obtain some sort of generalization of Theorem 1.5 to uncountable κ 's: this is essentially because with this alternative approach we will be able to associate to each κ -Souslin quasi order $R = p[T]$ a sentence $\bar{\sigma}_T$ in the *bounded* logic $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ still having the property that $R \sim \sqsubset_{\bar{\sigma}_T}^\kappa$, thus further obtaining that when κ is regular $\text{Mod}_{\bar{\sigma}_T}^\kappa$ is a $\kappa + 1$ -Borel subset of $\text{Mod}_{\bar{\mathcal{L}}}^\kappa$ by Corollary 8.10(b).

The alternative construction we are going to provide consists of the following steps:

- we expand the \mathcal{L} -structure $\mathbb{G}_0 \in \text{CT}_\kappa$ defined in Section 10.1 to a so-called *ordered* combinatorial tree $\bar{\mathbb{G}}_0$ by interpreting the new symbol \trianglelefteq in the extended language $\bar{\mathcal{L}} = \{\mathbf{E}, \trianglelefteq\}$ as a well-founded order on (the nodes of) \mathbb{G}_0 ;
- given $S \in \text{Tr}(2 \times \kappa)$, we directly add to \mathbb{G}_0 some forks as in Section 10.2 (without first enlarging \mathbb{G}_0 to an analogue of \mathbb{G}_1), leaving the interpretation of \trianglelefteq unchanged.

Let us now fix some further notation concerning the $\bar{\mathcal{L}}$ -structures that we are going to consider. Following [FMR11], a combinatorial tree with a supplementary transitive relation defined on a subset of its set of vertices will be called an **ordered combinatorial tree**. This is a bit of a misnomer, since this extra relation need not be an ordering, although in what follows this will always be the case. To formalize this concept, consider the expansion $\bar{\mathcal{L}} = \{\mathbf{E}, \trianglelefteq\}$ of the graph language \mathcal{L} (see page 73) with \trianglelefteq a binary relational symbol. Then $\text{OCT}_\kappa \subseteq \text{Mod}_{\bar{\mathcal{L}}}^\kappa$ is the collection of all ordered combinatorial trees of size κ (up to isomorphism), i.e. the set of all $X = \langle \kappa; \mathbf{E}^X, \trianglelefteq^X \rangle$ such that $\langle \kappa; \mathbf{E}^X \rangle$ is a combinatorial tree and \trianglelefteq^X is a transitive relation. Formally, $\text{OCT}_\kappa := \text{Mod}_{\sigma_{\text{OCT}}}^\kappa$, where σ_{OCT} is the following $\bar{\mathcal{L}}_{\omega_1\omega}$ -sentence axiomatizing ordered combinatorial trees:

$$(\sigma_{\text{OCT}}) \quad \sigma_{\text{CT}} \wedge \forall v_0 \forall v_1 \forall v_2 (v_0 \trianglelefteq v_1 \wedge v_1 \trianglelefteq v_2 \Rightarrow v_0 \trianglelefteq v_2).$$

Similarly to the case of combinatorial trees, in order to simplify the notation we will abbreviate the embeddability relation $\sqsubset \upharpoonright \text{Mod}_{\sigma_{\text{OCT}}}^\kappa$ and the isomorphism relation $\cong \upharpoonright \text{Mod}_{\sigma_{\text{OCT}}}^\kappa$ (see page 55) with $\sqsubset_{\text{OCT}}^\kappa$ and $\cong_{\text{OCT}}^\kappa$, respectively.

13.1. Completeness. To begin with, we slightly modify the construction presented in Section 10 by first adding a well-founded order on the vertices of the combinatorial tree \mathbb{G}_0

from Definition 10.2. Such order is essentially obtained using in the obvious way the bijection $\langle\!\langle\cdot\rangle\!\rangle: {}^{<\omega}\text{Ord} \rightarrow \text{Ord}$ from (2.2). More precisely, define

$$(13.1) \quad \# : {}^{<\omega}\kappa \uplus \{s^- \mid \emptyset \neq s \in {}^{<\omega}\kappa\} \rightarrow \kappa$$

by letting

$$\begin{aligned} \#(s) &:= 2\langle\!\langle s \rangle\!\rangle \\ \#(s^-) &:= \begin{cases} 2\langle\!\langle s \rangle\!\rangle - 1 & \text{if } \langle\!\langle s \rangle\!\rangle \in \omega, \\ 2\langle\!\langle s \rangle\!\rangle + 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Letting $s_\alpha := \#^{-1}(\alpha)$, one gets that $s_\alpha \in {}^{<\omega}\kappa$ (i.e. s_α is not of the form s^-) if and only if α is even. The ordered combinatorial tree $\bar{\mathbb{G}}_0$ is then the expansion of the structure $\mathbb{G}_0 = \langle G_0; \mathbf{E}^{G_0} \rangle$ obtained by interpreting \trianglelefteq as the well-order induced on G_0 by the above map $\#$. Formally:

DEFINITION 13.1. $\bar{\mathbb{G}}_0$ is the $\bar{\mathcal{L}}$ -structure on

$$\bar{G}_0 := G_0 = {}^{<\omega}\kappa \uplus \{s^- \mid \emptyset \neq s \in {}^{<\omega}\kappa\}$$

with edge relation $\mathbf{E}^{\bar{\mathbb{G}}_0}$ defined by

$$a \mathbf{E}^{\bar{\mathbb{G}}_0} b \Leftrightarrow a \mathbf{E}^{G_0} b$$

and order relation $\trianglelefteq^{\bar{\mathbb{G}}_0}$ defined by

$$a \trianglelefteq^{\bar{\mathbb{G}}_0} b \Leftrightarrow \#(a) \leq \#(b),$$

where $a, b \in \bar{G}_0$, \mathbf{E}^{G_0} is as in Definition 10.2, and $\#$ is the map from (13.1).

Notice that $\trianglelefteq^{\bar{\mathbb{G}}_0}$ is isomorphic to κ , so $\bar{\mathbb{G}}_0$ is rigid. We next obtain the structures $\bar{\mathbb{G}}_S$ (for $S \in \text{Tr}(2 \times \kappa)$) by joining $\bar{\mathbb{G}}_0$ and the forks $F_{u,s}$ defined on page 78 (for every $(u, s) \in S$) via the identification of $s \in \bar{G}_0$ with the vertex (u, s, \emptyset) of $F_{u,s}$ — note that this construction is exactly the same described in Section 10.2 except that now we are avoiding the enlargement to the structure \mathbb{G}_1 .

DEFINITION 13.2. Let $S \in \text{Tr}(2 \times \kappa)$. Then $\bar{\mathbb{G}}_S$ is the $\bar{\mathcal{L}}$ -structure on

$$\bar{G}_S := \bar{G}_0 \uplus \{(u, s, w) \mid (u, s) \in S \wedge (u, s, w) \in F_{u,s} \wedge w \neq \emptyset\}$$

with edge relation $\mathbf{E}^{\bar{\mathbb{G}}_S}$ defined by

$$a \mathbf{E}^{\bar{\mathbb{G}}_S} b \Leftrightarrow a \mathbf{E}^{G_S} b$$

and order relation $\trianglelefteq^{\bar{\mathbb{G}}_S}$ defined by

$$a \trianglelefteq^{\bar{\mathbb{G}}_S} b \Leftrightarrow a, b \in \bar{G}_0 \wedge a \trianglelefteq^{\bar{\mathbb{G}}_0} b,$$

where $a, b \in \bar{G}_S$ and \mathbf{E}^{G_S} is the edge relation on G_S defined in Section 10.2.

Notice that the above definition is well-given because $\bar{G}_S \subseteq G_S$: in fact, using the notation introduced in (10.7a)–(10.7g) and Remark 10.7

$$\bar{G}_S = \text{Seq} \uplus \text{Seq}^- \uplus \bigcup \{F'_{u,s} \mid (u, s) \in S\} = G_S \setminus (\widehat{\text{Seq}} \cup \mathbf{U}).$$

Each structure \mathbb{G}_S is then further identified in a canonical way with its copy on κ (which is thus an element of OCT_κ), using the bijections $\langle\!\langle\cdot\rangle\!\rangle$, $\langle\cdot, \cdot\rangle$, and e_u from (2.2), (2.1), and (10.6) in the obvious way.

Finally, given an infinite cardinal κ and $T \in \mathbb{T}_\kappa$ we let \bar{f}_T be the variant of the function f_T from (11.5) obtained by composing the map $\Sigma_T: {}^\omega 2 \rightarrow \text{Tr}(2 \times \kappa)$ from Definition 11.6 with the modified map $S \mapsto \bar{\mathbb{G}}_S$, that is:

$$(13.2) \quad \bar{f}_T: {}^\omega 2 \rightarrow \text{OCT}_\kappa, \quad x \mapsto \bar{\mathbb{G}}_{\Sigma_T(x)}.$$

THEOREM 13.3. *Let κ be an infinite cardinal and $T \in \mathbb{T}_\kappa$. If $R = p[T]$ is a quasi-order, then the map f_T defined in (13.2) is such that:*

- (a) \bar{f}_T reduces R to the embeddability relation $\sqsubseteq_{\text{OCT}}^\kappa$;
- (b) \bar{f}_T reduces $=$ on ${}^\omega 2$ to the isomorphism relation $\cong_{\text{OCT}}^\kappa$.

In particular, $\sqsubseteq_{\text{OCT}}^\kappa$ is complete for κ -Souslin quasi-orders.

PROOF. For part (a) just check that the same proof of Theorem 11.8(a) goes through also with the new construction with the following minor modifications:

- in the forward direction, to get the embedding between $\bar{f}_T(x) = \bar{\mathbb{G}}_{\Sigma_T(x)}$ and $\bar{f}_T(y) = \bar{\mathbb{G}}_{\Sigma_T(y)}$ when $x R y$ we of course just consider the restriction of i to $\bar{G}_{\Sigma_T(x)} \subseteq G_{\Sigma_T(x)}$ instead of the full embedding $i: \mathbb{G}_{\Sigma_T(x)} \rightarrow \mathbb{G}_{\Sigma_T(y)}$ described in the original proof;
- for the backward direction, we use Remark 11.9 and the fact that $\bar{\mathbb{G}}_{\Sigma_T(x)}$ and $\bar{\mathbb{G}}_{\Sigma_T(y)}$ are just $\bar{\mathcal{L}}$ -expansions of $\mathbb{G}_{\Sigma_T(x)} \upharpoonright (G_0 \cup F(\mathbb{G}_{\Sigma_T(x)}))$ and $\mathbb{G}_{\Sigma_T(y)} \upharpoonright (G_0 \cup F(\mathbb{G}_{\Sigma_T(y)}))$, respectively.

For part (b), fix an isomorphism j between $\bar{f}_T(x) = \bar{\mathbb{G}}_{\Sigma_T(x)}$ and $\bar{f}_T(y) = \bar{\mathbb{G}}_{\Sigma_T(y)}$. Then $j(\bar{G}_0) = \bar{G}_0$ because the vertices in $\bar{G}_0 \subseteq \bar{G}_{\Sigma_T(x)}, \bar{G}_{\Sigma_T(y)}$ are the unique ones which are in the domain of the orders $\leq^{\bar{\mathbb{G}}_{\Sigma_T(x)}}$ and $\leq^{\bar{\mathbb{G}}_{\Sigma_T(y)}}$, respectively. Since the relational symbol \leq is interpreted in both $\bar{\mathbb{G}}_{\Sigma_T(x)}$ and $\bar{\mathbb{G}}_{\Sigma_T(y)}$ as the same well-order on \bar{G}_0 (of the same order type κ), we get that $j \upharpoonright \bar{G}_0$ is the identity function. Arguing as in Theorem 11.8(b), we then get that this implies $\Sigma_T(x) = \Sigma_T(y)$, and hence that $x = y$ by injectivity of Σ_T (Lemma 11.7(c)). \square

13.2. Invariant universality. Henceforth we again fix an *uncountable* cardinal κ , and show that $\sqsubseteq_{\text{OCT}}^\kappa$ is invariantly universal for κ -Souslin quasi-orders. Following Section 12, we will first provide an $\bar{\mathcal{L}}_{\kappa+\kappa}$ -formula $\bar{\Psi}$ describing the common parts of the $\bar{\mathbb{G}}_S$ (for $S \in \text{Tr}(2 \times \kappa)$), and then classify the structures in $\text{Mod}_{\bar{\Psi}}^\kappa$ up to isomorphism using the elements of ${}^{<\omega} 2^{\times \kappa}$ as invariants. The main improvement of the new construction is that $\bar{\Psi}$ will be a sentence in the bounded logic $\bar{\mathcal{L}}_{\kappa+\kappa}^b$, so that when κ is regular $\text{Mod}_{\bar{\Psi}}^\kappa$ is an effective $\kappa + 1$ -Borel set by Corollary 8.10(b). As in Section 12, to simplify the presentation we will freely use metavariables and consider (infinitary) conjunctions and disjunctions over (canonically) well-orderable sets of formulæ of size $\leq \kappa$.

Given an ordinal $\alpha < \kappa$ and a sequence of variables $\langle x_\beta \mid \beta < \alpha \rangle$, let $\bar{\rho}(\langle x_\beta \mid \beta < \alpha \rangle)$ be the $\bar{\mathcal{L}}_{|\alpha|+\omega}^0$ -formula

$$(\bar{\rho}) \quad \bigwedge_{\beta < \gamma < \alpha} (x_\beta \neq x_\gamma \wedge x_\beta \leq x_\gamma \wedge \neg(x_\gamma \leq x_\beta)).$$

Notice that for any $\bar{\mathcal{L}}$ -structure X with domain κ and any assignment $s \in {}^\alpha \kappa$, $X \models \bar{\rho}[s]$ if and only if s is injective and $\text{ran}(s)$ is well-ordered by \leq^X in order type α . Now let $\bar{\Phi}_1$ be the $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -sentence given by the conjunction of the following:

- (i) $\forall x, y [\exists z (x \leq z \vee z \leq x) \wedge \exists z (y \leq z \vee z \leq y) \Rightarrow x \leq y \vee y \leq x]$;
- (ii) $\forall x, y, z (x \leq y \wedge y \leq z \Rightarrow x \leq z)$;
- (iii) $\forall x, y (x \leq y \wedge y \leq x \Rightarrow x \simeq y)$;
- (iv) $\neg \exists \langle x_n \mid n < \omega \rangle [\bigwedge_{n < m < \omega} (x_m \neq x_n \wedge x_m \leq x_n)]$;
- (v) $\bigwedge_{\alpha < \kappa} \exists \langle x_\beta \mid \beta < \alpha \rangle \bar{\rho}(\langle x_\beta \mid \beta < \alpha \rangle)$;
- (vi) $\neg \exists x \bigwedge_{\alpha < \kappa} \exists \langle y_\beta \mid \beta < \alpha \rangle \bar{\rho}(\langle y_\beta \mid \beta < \alpha \rangle \frown x)$.

If X is a structure as above and satisfies (i)–(iv) then \leq^X is a well-order.³⁵ If moreover it satisfies (v) the length of \leq^X is $\geq \kappa$, and if it satisfies (vi) the length is $< \kappa + 1$. Therefore if

³⁵We don't need to assume DC to have the equivalence between “well-foundedness” and “absence of descending chains” because in our situation the domain of the structure X is always assumed to be the cardinal κ , which carries a natural well-order.

$X \models \bar{\Phi}_1$ then \trianglelefteq^X is a well-order of length κ , possibly defined just on a subset of the domain of X .

For every ordinal $\alpha < \kappa$ let $\bar{\rho}_\alpha(x)$ be the $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -formula

$$(\bar{\rho}_\alpha) \quad \exists \langle x_\beta \mid \beta < \alpha \rangle \bar{\rho}(\langle x_\beta \mid \beta < \alpha \rangle \frown x) \wedge \neg \exists \langle x_\beta \mid \beta < \alpha + 1 \rangle \bar{\rho}(\langle x_\beta \mid \beta < \alpha + 1 \rangle \frown x).$$

stating that x is the α -th element in the order \trianglelefteq , at least whenever \trianglelefteq is (interpreted in) a well-order of length greater than α . Given ordinals $\alpha, \beta < \kappa$ and two variables x_0, x_1 , let $\bar{\Psi}_{\alpha,\beta}(x_0, x_1)$ be the formula $x_0 \mathbf{E} x_1$ if $s_\alpha \mathbf{E}^{\bar{\mathbb{G}}_0} s_\beta$ or $\neg(x_0 \mathbf{E} x_1)$ otherwise, where $s_\alpha := \#^{-1}(\alpha)$ and $\#$ is as in (13.1). Let $\bar{\Phi}_2$ be the $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -sentence:

$$(\bar{\Phi}_2) \quad \bigwedge_{\alpha < \beta < \kappa} \forall x, y (\bar{\rho}_\alpha(x) \wedge \bar{\rho}_\beta(y) \Rightarrow \bar{\Psi}_{\alpha,\beta}(x, y) \wedge \bar{\Psi}_{\beta,\alpha}(y, x)).$$

Notice that if an $\bar{\mathcal{L}}$ -structure X satisfies $\bar{\Phi}_1 \wedge \bar{\Phi}_2$, then its restriction to the field of \trianglelefteq^X is isomorphic to $\bar{\mathbb{G}}_0$, and moreover such an isomorphism is unique by the rigidity of $\bar{\mathbb{G}}_0$.

For every $u \in {}^{<\omega}2$, let \bar{F}_u be the $\bar{\mathcal{L}}$ -expansion of F_u in which $\emptyset \trianglelefteq^{\bar{F}_u} \emptyset$ and no other $\trianglelefteq^{\bar{F}_u}$ -relation holds. We call \emptyset the **root** of \bar{F}_u . Notice that \bar{F}_u still satisfies analogues of (10.5a)–(10.5c). In particular, for any $\bar{\mathcal{L}}$ -structure $X \cong \bar{F}_u$ the isomorphism between X and \bar{F}_u is unique, and we can thus unambiguously call root of X the unique element of X which is mapped by such isomorphism to the root \emptyset of \bar{F}_u .

Given $n, m \in \omega$, $u \in {}^{<\omega}2$, and two variables x_0, x_1 , let $\bar{\chi}_{n,m}^u(x_0, x_1)$ be the $\bar{\mathcal{L}}$ -formula $x_0 \mathbf{E} x_1$ if $e_u^{-1}(n) \mathbf{E}^{F_u} e_u^{-1}(m)$ and $\neg(x_0 \mathbf{E} x_1)$ otherwise, where e_u is the bijection of (10.6). Then $\bar{\chi}_u(\langle x_n \mid n \in \omega \rangle)$ denotes the $\bar{\mathcal{L}}_{\omega_1\omega}^0$ -formula

$$(\bar{\chi}_u) \quad \bigwedge_{n < m < \omega} (x_n \neq x_m) \wedge \bigwedge_{\substack{n, m < \omega \\ n > 0}} \bar{\chi}_{n,m}^u(x_n, x_m) \\ \wedge (x_0 \trianglelefteq x_0) \wedge \bigwedge_{\substack{n, m < \omega \\ n > 0}} (\neg(x_n \trianglelefteq x_m) \wedge \neg(x_m \trianglelefteq x_n)).$$

Notice that for any $\bar{\mathcal{L}}$ -structure X and any $\langle a_n \mid n \in \omega \rangle \in {}^\omega X$,

$$(13.3) \quad X \models \bar{\chi}_u[\langle a_n \mid n \in \omega \rangle] \Leftrightarrow \text{the substructure of } X \text{ with domain } \{a_n \mid n \in \omega\} \\ \text{is isomorphic to } \bar{F}_u \text{ (via the map } a_n \mapsto e_u^{-1}(n)).$$

Moreover such isomorphism must again be unique by (10.5b). Let now $\bar{\Phi}_3$ be the following $\bar{\mathcal{L}}_{\omega_1\omega_1}^b$ -sentence:

$$(\bar{\Phi}_3) \quad \forall x \left[\neg(x \trianglelefteq x) \Rightarrow \bigvee_{u \in {}^{<\omega}2} \exists \langle y_n \mid n < \omega \rangle \left(\bar{\chi}_u(\langle y_n \mid n < \omega \rangle) \wedge \bigvee_{0 \neq i < \omega} x \simeq y_i \right) \right].$$

This means that if an $\bar{\mathcal{L}}$ -structure X satisfies $\bar{\Phi}_1 \wedge \bar{\Phi}_3$, then each element of X which is not in the field of \trianglelefteq^X belongs to some substructure which is isomorphic to \bar{F}_u (for some $u \in {}^{<\omega}2$).

Let now $\bar{\Phi}_4$ be the following $\bar{\mathcal{L}}_{\omega_1\omega_1}^b$ -sentence:

$$(\bar{\Phi}_4) \quad \forall x \forall \langle y_n \mid n < \omega \rangle \left[\bigvee_{u \in {}^{<\omega}2} \bar{\chi}_u(\langle y_n \mid n < \omega \rangle) \wedge \bigwedge_{n < \omega} (x \neq y_n) \right. \\ \left. \Rightarrow \bigwedge_{0 \neq n < \omega} (\neg(x \mathbf{E} y_n) \wedge \neg(x \trianglelefteq y_n) \wedge \neg(y_n \trianglelefteq x)) \right].$$

Note that if X is an $\bar{\mathcal{L}}$ -structure such that $X \models \sigma_{\text{OCT}} \wedge \bar{\Phi}_4$, then any of its substructures which is isomorphic to some \bar{F}_u is “isolated” from the rest of X , meaning that each element of such

substructure (except for its root) is neither \mathbf{E}^X -related nor \leq^X -related to any other element of X which does not belong to said substructure.

Let $\bar{\Phi}_5$ be the following $\bar{\mathcal{L}}_{\omega_1\omega_1}^b$ -sentence:

$$(\bar{\Phi}_5) \quad \forall \langle x_n \mid n < \omega \rangle \forall \langle y_m \mid m < \omega \rangle \left[\bigvee_{u \in <\omega_2} \bar{\chi}_u(\langle x_n \mid n < \omega \rangle) \wedge \bigvee_{v \in <\omega_2} \bar{\chi}_v(\langle y_m \mid m < \omega \rangle) \right. \\ \left. \Rightarrow \bigwedge_{n < \omega} (x_n \simeq y_n) \vee \bigwedge_{\substack{n, m < \omega \\ (n, m) \neq (0, 0)}} (x_n \neq y_m) \right].$$

This means that if an $\bar{\mathcal{L}}$ -structure X satisfies $\bar{\Phi}_5$, then any two of its substructures which are isomorphic to, say, \bar{F}_u and \bar{F}_v , either coincide or else share at most the same root.

Finally, let $\bar{\Phi}_6$ be the following $\bar{\mathcal{L}}_{\omega_1\omega_1}^b$ -sentence:

$$(\bar{\Phi}_6) \quad \forall \langle x_n \mid n < \omega \rangle \forall \langle y_m \mid m < \omega \rangle \left[\bigvee_{u \in <\omega_2} (\bar{\chi}_u(\langle x_n \mid n \in \omega \rangle) \wedge \bar{\chi}_u(\langle y_m \mid m \in \omega \rangle)) \right. \\ \left. \Rightarrow \bigwedge_{n \in \omega} (x_n \simeq y_n) \vee \bigwedge_{n, m \in \omega} (x_n \neq y_m) \right].$$

This means that if $X \models \bar{\Phi}_6$, where X is an $\bar{\mathcal{L}}$ -structure, then any two of its substructures which are isomorphic to the same \bar{F}_u , either they coincide or else they are completely disjoint.

Let finally $\bar{\Psi}$ be the $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -sentence

$$(13.4) \quad \sigma_{\text{OCT}} \wedge \bar{\Phi}_1 \wedge \bar{\Phi}_2 \wedge \bar{\Phi}_3 \wedge \bar{\Phi}_4 \wedge \bar{\Phi}_5 \wedge \bar{\Phi}_6.$$

It is easy to check that:

LEMMA 13.4. *Let $T \in \mathbb{T}_\kappa$ and \bar{f}_T be as in (13.2). Then $\text{ran}(\bar{f}_T) \subseteq \text{Mod}_{\bar{\Psi}}^\kappa$.*

We now classify again the structures in $\text{Mod}_{\bar{\Psi}}^\kappa$ using the elements of ${}^{<\omega}2^{\times \kappa 2}$ as invariant. For $X \in \text{Mod}_{\bar{\Psi}}^\kappa$, define $\bar{g}(X): {}^{<\omega}2^{\times \kappa} \rightarrow 2$ by letting $\bar{g}(X)(u, \alpha) = 1$ if and only if there is a substructure of X isomorphic to \bar{F}_u whose root is the α -th element with respect to the order \leq^X . Formally,

$$(13.5) \quad \bar{g}: \text{Mod}_{\bar{\Psi}}^\kappa \rightarrow {}^{<\omega}2^{\times \kappa 2}, \quad \bar{g}(X)(u, \alpha) = 1 \Leftrightarrow X \models \bar{\sigma}_{u, \alpha},$$

where $\bar{\sigma}_{u, \alpha}$ is the $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -sentence

$$(\bar{\sigma}_{u, \alpha}) \quad \exists x [\bar{\rho}_\alpha(x) \wedge \exists \langle z_n \mid n < \omega \rangle (\bar{\chi}_u(\langle z_n \mid n < \omega \rangle) \wedge x \simeq z_0)].$$

The following proposition is the analogue in the new context of Proposition 12.7, and its proof is quite similar to (but simpler than) the original one. However, for the reader's convenience we fully reprove it here because such classification result lies at the core of the proof of the invariant universality of $\sqsubset_{\text{OCT}}^\kappa$.

PROPOSITION 13.5. *The map \bar{g} defined in (13.5) reduces \cong to $=$.*

PROOF. Let $X, Y \in \text{Mod}_{\bar{\Psi}}^\kappa$ be isomorphic. Then for every $(u, \alpha) \in {}^{<\omega}2^{\times \kappa}$, $X \models \bar{\sigma}_{u, \alpha} \Leftrightarrow Y \models \bar{\sigma}_{u, \alpha}$, whence $\bar{g}(X) = \bar{g}(Y)$ by (13.5).

Conversely, assume $\bar{g}(X) = \bar{g}(Y)$ for some $X, Y \in \text{Mod}_{\bar{\Psi}}^\kappa$. Since X and Y are models of $\bar{\Psi}$ they satisfy σ_{OCT} and $\bar{\Phi}_1, \dots, \bar{\Phi}_6$. Let X' and Y' be the substructures of X and Y whose domains are the fields of the orderings \leq^X and \leq^Y , respectively. As X and Y satisfy $\bar{\Phi}_1 \wedge \bar{\Phi}_2$, then $X' \cong Y'$ via a (unique) isomorphism $\iota_{X', Y'}$. We will now extend $\iota_{X', Y'}$ to an isomorphism

$$\iota_{X, Y}: X \rightarrow Y.$$

Let $a \in X \setminus X'$. Since $X \models \bar{\Phi}_3$, there are $\langle a_n \mid n < \omega \rangle \in {}^\omega X$ and $0 \neq \bar{i} < \omega$ such that $a = a_{\bar{i}}$ and $X \models \bar{\chi}_u[\langle a_n \mid n < \omega \rangle]$ for some $u \in {}^{<\omega}2$. In particular, a_0 is in the field X' of \trianglelefteq^X , which is a well-order of length κ since $X \models \bar{\Phi}_1$. Let $\alpha < \kappa$ be such that a_0 is the α -th element in this order: then by definition of \bar{g} we have $\bar{g}(X)(u, \alpha) = 1$, so $\bar{g}(Y)(u, \alpha) = 1$ by case assumption. Let $b_0 \in Y'$ be the α -th element with respect to the order \trianglelefteq^Y — b_0 is well-defined because $Y \models \bar{\Phi}_1$. Choose $\langle b_1, b_2, \dots \rangle \in {}^\omega Y$ such that $Y \models \bar{\chi}_u[\langle b_n \mid n < \omega \rangle]$, and set $\iota_{X,Y}(a) := b_{\bar{i}}$. The definition of $\iota_{X,Y}(a)$ seems to depend on the choice of the b_n 's, but the next claim shows that this is not the case.

CLAIM 13.5.1. *Suppose $\alpha < \kappa$, $u \in {}^{<\omega}2$, and $\langle a_n \mid n < \omega \rangle$ is a sequence of elements of X such that a_0 is the α -th element in \trianglelefteq^X and $X \models \bar{\chi}_u[\langle a_n \mid n < \omega \rangle]$. Then there is a unique sequence $\langle b_n \mid n < \omega \rangle$ of elements of Y such that b_0 is the α -th element in \trianglelefteq^Y and $Y \models \bar{\chi}_u[\langle b_n \mid n < \omega \rangle]$. Therefore $\iota_{X,Y}(a_n) = b_n$ for every $n < \omega$.*

PROOF OF THE CLAIM. Given two sequences $\langle b_n \mid n < \omega \rangle$ and $\langle b'_n \mid n < \omega \rangle$ as in the conclusion of the claim, we have that $b_0 = b'_0$ and both $Y \models \bar{\chi}_u[\langle b_n \mid n < \omega \rangle]$ and $Y \models \bar{\chi}_u[\langle b'_n \mid n < \omega \rangle]$. Since $Y \models \bar{\Phi}_6$, it follows that $b_n = b'_n$ for all $n < \omega$. \square

CLAIM 13.5.2. *$\iota_{X,Y}$ is injective.*

PROOF OF THE CLAIM. Let $a, a' \in X$ be distinct. If either a or a' belongs to X' , then $\iota_{X,Y}(a) \neq \iota_{X,Y}(a')$ because $\iota_{X,Y} \upharpoonright X' = \iota_{X',Y'}$ is a bijection between X' and Y' , and $\iota_{X,Y}(X \setminus X') \subseteq Y \setminus Y'$ by construction. So we can assume $a, a' \notin X'$.

Assume towards a contradiction that $\iota_{X,Y}(a) = \iota_{X,Y}(a')$. Since $X \models \bar{\Phi}_1 \wedge \bar{\Phi}_3$, let $\alpha, \beta < \kappa$, $u, v \in {}^{<\omega}2$, $\langle a_n \mid n < \omega \rangle, \langle a'_n \mid n < \omega \rangle \in {}^\omega X$, and $0 < \bar{i}, \bar{j} < \omega$ be such that a_0 and a'_0 are, respectively, the α -th and β -th elements of \trianglelefteq^X , $X \models \bar{\chi}_u[\langle a_n \mid n < \omega \rangle]$, $X \models \bar{\chi}_v[\langle a'_n \mid n < \omega \rangle]$, $a = a_{\bar{i}}$, and $a' = a'_{\bar{j}}$. By Claim 13.5.1 let $\langle b_n \mid n < \omega \rangle$ and $\langle b'_n \mid n < \omega \rangle$ be the unique sequences such that b_0 and b'_0 are, respectively, the α -th and β -th elements of \trianglelefteq^Y , $Y \models \bar{\chi}_u[\langle b_n \mid n < \omega \rangle]$, and $Y \models \bar{\chi}_v[\langle b'_n \mid n < \omega \rangle]$. Then $b_n = \iota_{X,Y}(a_n)$ and $b'_n = \iota_{X,Y}(a'_n)$ for every n , so that $\iota_{X,Y}(a) = b_{\bar{i}}$ and $\iota_{X,Y}(a') = b'_{\bar{j}}$. Since $Y \models \bar{\Phi}_5$ and

$$(13.6) \quad b_{\bar{i}} = \iota_{X,Y}(a) = \iota_{X,Y}(a') = b'_{\bar{j}},$$

it follows that

$$(13.7) \quad \forall n (b_n = b'_n).$$

Let Z be the substructure of Y with domain $\{b_n \mid n \in \omega\}$. Since $Y \models (\bar{\chi}_u \wedge \bar{\chi}_v)[\langle b_n \mid n < \omega \rangle]$, then $\bar{F}_u \cong Z \cong \bar{F}_v$ by (13.3). Thus $u = v$ by (the analogue of) (10.5c).

Notice that $a_0, a'_0 \in X'$ and $b_0, b'_0 \in Y'$, thus $b_0 = b'_0$ by (13.7), and hence $a_0 = a'_0$ as $\iota_{X,Y} \upharpoonright X' = \iota_{X',Y'}$ is a bijection between X' and Y' . Moreover, $Y \models \bar{\chi}_u[\langle b_n \mid n < \omega \rangle]$ implies that the b_n 's are all distinct. Therefore $\bar{i} = \bar{j}$ because $b_{\bar{i}} = b_{\bar{j}}$ by (13.6) and (13.7). Since $u = v$, it follows that $X \models \bar{\chi}_u[\langle a_n \mid n < \omega \rangle]$ and $X \models \bar{\chi}_u[\langle a'_n \mid n < \omega \rangle]$. Therefore, $a_0 = a'_0$ and $X \models \bar{\Phi}_6$ imply $\forall n (a_n = a'_n)$, whence $a = a_{\bar{i}} = a'_{\bar{i}} = a'_j = a'$, a contradiction. \square

CLAIM 13.5.3. *$\iota_{X,Y}$ is surjective and $\iota_{Y,X} = \iota_{X,Y}^{-1}$.*

PROOF OF THE CLAIM. First notice that $\iota_{Y,X} \upharpoonright Y' = \iota_{Y',X'}$ is an isomorphism between Y' and X' and is the inverse of $\iota_{X',Y'} = \iota_{X,Y} \upharpoonright X'$ (by the unicity of the isomorphism between X' and Y'). Let now $b \in Y \setminus Y'$. Since $Y \models \bar{\Phi}_1 \wedge \bar{\Phi}_3$, there are $\alpha < \kappa$, $u \in {}^{<\omega}2$, $\langle b_n \mid n \in \omega \rangle \in {}^\omega Y$, and $0 \neq \bar{i} < \omega$ such that b_0 is the α -th element in the order \trianglelefteq^Y , $Y \models \bar{\chi}_u[\langle b_n \mid n < \omega \rangle]$, and $b = b_{\bar{i}}$. Applying Claim 13.5.1 with the role of X and Y interchanged, there is a unique sequence $\langle a_n \mid n \in \omega \rangle \in {}^\omega X$ such that a_0 is the α -th element in \trianglelefteq^X and $X \models \bar{\chi}_u[\langle a_n \mid n < \omega \rangle]$. Then $\iota_{Y,X}(b) = a_{\bar{i}}$ by definition. Using Claim 13.5.1, it is easy to check that $\iota_{X,Y}(a_{\bar{i}}) = b$, so we are done. \square

We have shown that $\iota_{X,Y}: X \rightarrow Y$ is a bijection. It remains to be proved that it is also an isomorphism. Since $\iota_{X,Y}^{-1} = \iota_{Y,X}$, it is enough to show just one direction of each equivalence, i.e. that for every $a, a' \in X$

$$\begin{aligned} a \mathbf{E}^X a' &\Rightarrow \iota_{X,Y}(a) \mathbf{E}^Y \iota_{X,Y}(a') \\ a \leq^X a' &\Rightarrow \iota_{X,Y}(a) \leq^Y \iota_{X,Y}(a'). \end{aligned}$$

If $a, a' \in X'$ the result follows from the fact that $\iota_{X,Y} \upharpoonright X' = \iota_{X',Y'}$ is an isomorphism. In particular, since $a \leq^X a'$ implies $a, a' \in X'$, the second implication holds. Thus it is enough to focus on the first implication and assume, without loss of generality, that $a \notin X'$ (both \mathbf{E}^X and \mathbf{E}^Y are symmetric relations by $X \models \sigma_{\text{OCT}}$ and $Y \models \sigma_{\text{OCT}}$). Since $X \models \bar{\Phi}_1 \wedge \bar{\Phi}_3$, there are $\alpha < \kappa$, $u \in {}^{<\omega}2$, $\langle a_n \mid n \in \omega \rangle \in {}^\omega X$, and $0 \neq \bar{i} < \omega$ such that a_0 is the α -th element in \leq^X , $X \models \bar{\chi}_u[\langle a_n \mid n < \omega \rangle]$, and $a = a_{\bar{i}}$. If $a \mathbf{E}^X a'$, then $X \models \sigma_{\text{OCT}} \wedge \bar{\Phi}_4$ implies $a' = a_{\bar{j}}$ for some $\bar{j} \neq \bar{i}$. By Claim 13.5.1, let $\langle b_n \mid n \in \omega \rangle$ be the unique sequence such that b_0 is the α -th element in \leq^Y and $Y \models \bar{\chi}_u[\langle b_n \mid n < \omega \rangle]$, so that $b_n = \iota_{X,Y}(a_n)$ for all $n \in \omega$ by definition. Since $a_{\bar{i}} = a \mathbf{E}^X a' = a_{\bar{j}}$ and $X \models \bar{\chi}_u[\langle a_n \mid n < \omega \rangle]$ imply that the subformula $\bar{\chi}_{\bar{i},\bar{j}}^u(x_{\bar{i}}, x_{\bar{j}})$ of $\bar{\chi}_u(\langle x_n \mid n < \omega \rangle)$ is $x_{\bar{i}} \mathbf{E} x_{\bar{j}}$, and since $Y \models \bar{\chi}_u[\langle b_n \mid n < \omega \rangle]$, we get $\iota_{X,Y}(a) = b_{\bar{i}} \mathbf{E}^Y b_{\bar{j}} = \iota_{X,Y}(a')$. \square

Endow ${}^{<\omega}2 \times {}^\kappa 2$ with the product topology τ_p (see Section 7.2.1 and, in particular, Example 7.8(A)). The following is the analogue of Lemma 12.9 and can be proven in the same way (just replace the $\mathcal{L}_{\kappa+\kappa}^b$ -sentence $\sigma_{u,\alpha}$ with the $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -sentence $\bar{\sigma}_{u,\alpha}$).

LEMMA 13.6. *For every (τ_p) -open set $V \subseteq {}^{<\omega}2 \times {}^\kappa 2$ there is an $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -sentence $\bar{\sigma}_V$ such that $g^{-1}(V) = \text{Mod}_{\bar{\sigma}_V \wedge \bar{\Psi}}^\kappa$.*

The following lemma, which corresponds to Lemma 12.10, requires instead a slightly different argument due to the different coding we used.

LEMMA 13.7. *Let $T \in \mathbb{T}_\kappa$ and \bar{f}_T be the map defined in (13.2). Then the map $\bar{g} \circ \bar{f}_T: {}^\omega 2 \rightarrow {}^{<\omega}2 \times {}^\kappa 2$ is continuous.*

PROOF. Arguing as in the proof of Lemma 12.10, since $\mathcal{S} = \{\tilde{N}_{(u,\alpha),i}^A \mid (u,\alpha) \in {}^{<\omega}2 \times \kappa \wedge i = 0, 1\}$ is a subbasis for the product topology on ${}^A 2$ (where $A := {}^{<\omega}2 \times \kappa$), it is enough to check that $(\bar{g} \circ \bar{f}_T)^{-1}(\tilde{N}_{(u,\alpha),i}^A)$ is open in ${}^\omega 2$ for every $(u,\alpha) \in {}^{<\omega}2 \times \kappa$ and $i = 0, 1$. Fix $(u,\alpha) \in {}^{<\omega}2 \times \kappa$. If α is odd then $(\bar{g} \circ \bar{f}_T)^{-1}(\tilde{N}_{(u,\alpha),1}^A) = \emptyset$ and $(\bar{g} \circ \bar{f}_T)^{-1}(\tilde{N}_{(u,\alpha),0}^A) = {}^\omega 2$. If α is even let

$$A_1 := \left\{ v \in {}^{\text{lh } u} 2 \mid (u, v, s_\alpha) \in \bar{T} \right\},$$

where \bar{T} is the tree obtained from T as in (11.3) and $s_\alpha = \#^{-1}(\alpha)$, and let $A_0 := {}^{\text{lh } u} 2 \setminus A_1$. Then by definition of \bar{f}_T and \bar{g} we get $(\bar{g} \circ \bar{f}_T)^{-1}(\tilde{N}_{(u,\alpha),i}^A) = \bigcup_{v \in A_i} \mathbf{N}_v$. \square

Finally, the next corollaries are the counterparts of Corollaries 12.11 and 12.12: their proofs are obtained from the original ones by systematically replacing Lemmas 12.6, 12.9, and 12.10, Proposition 12.7, and Corollary 12.11, with Lemmas 13.4, 13.6, and 13.7, Proposition 13.5, and Corollary 13.8, respectively.

COROLLARY 13.8. *Let $T \in \mathbb{T}_\kappa$ and \bar{f}_T be as in (13.2). For every closed set $C \subseteq {}^\omega 2$, $(\bar{g} \circ \bar{f}_T)(C)$ is closed.*

COROLLARY 13.9. *Let $T \in \mathbb{T}_\kappa$ and \bar{f}_T be as in (13.2). There is an $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -sentence $\bar{\sigma}_T$ such that the closure under isomorphism of $\text{ran}(\bar{f}_T)$ is $\text{Mod}_{\bar{\sigma}_T}^\kappa$.*

We finally defined the “inverse” map of \bar{f}_T analogously to Definition 12.13.

DEFINITION 13.10. For $T \in \mathbb{T}_\kappa$, \bar{f}_T as in (13.2) and $\bar{\sigma}_T$ as in Corollary 13.9, let $\bar{h}_T: \text{Mod}_{\bar{\sigma}_T}^\kappa \rightarrow {}^\omega 2$ be defined by

$$(13.8) \quad \bar{h}_T(X) = x \Leftrightarrow \bar{f}_T(x) \cong X.$$

Note that the function \bar{h}_T is well-defined by Corollary 13.9 and Theorem 13.3. The following corollary is analogous to Corollary 12.14, and it completes the proof of Theorem 13.12: it can be proved with an argument similar to the original one (but replacing Theorem 11.8 with Theorem 13.3).

COROLLARY 13.11. Let $T \in \mathbb{T}_\kappa$, \bar{f}_T be as in (13.2), $\bar{\sigma}_T$ as in Corollary 13.9, and \bar{h}_T as in Definition 13.10. If $R = p[T]$ is a quasi-order then \bar{h}_T simultaneously reduces \sqsubseteq to R and \cong to $=$.

THEOREM 13.12. Let κ be an infinite cardinal and $T \in \mathbb{T}_\kappa$. If $R = p[T]$ is a quasi-order, then there are $\bar{\sigma}_T, \bar{f}_T, \bar{h}_T$ such that:

- (a) $\bar{\sigma}_T$ is an $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ sentence;
- (b) \bar{f}_T reduces R to $\sqsubseteq_{\bar{\sigma}_T}^\kappa$ and $=$ to $\cong_{\bar{\sigma}_T}^\kappa$;
- (c) \bar{h}_T reduces $\sqsubseteq_{\bar{\sigma}_T}^\kappa$ to R and $\cong_{\bar{\sigma}_T}^\kappa$ to $=$;
- (d) $\bar{h}_T \circ \bar{f}_T = \text{id}$ and $(\bar{f}_T \circ \bar{h}_T)(X) \cong X$ for every $X \in \text{Mod}_{\bar{\sigma}_T}^\kappa$.

In particular, $\sqsubseteq_{\text{OCT}}^\kappa$ is invariantly universal for κ -Souslin quasi-orders on ${}^\omega 2$ (equivalently, on any uncountable Polish or standard Borel space).

By Theorem 13.12(d) we further have:

COROLLARY 13.13. Let κ be an infinite cardinal and $T \in \mathbb{T}_\kappa$. If $R = p[T]$ is a quasi-order, then there is an $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -sentence $\bar{\sigma}_T$ such that the quotient orders of R and $\sqsubseteq_{\bar{\sigma}_T}^\kappa$ are isomorphic.

Theorem 13.12(a) should be contrasted with Theorem 12.15(a), in that it involves a formula belonging to the fragment $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ of $\bar{\mathcal{L}}_{\kappa+\kappa}$: this yields to the fact that by Corollary 8.10(b)

(13.9) if κ is regular, $\text{Mod}_{\bar{\sigma}_T}^\kappa$ is an effective $\kappa + 1$ -Borel subset of $\text{Mod}_{\bar{\mathcal{L}}}^\kappa$ (with respect to τ_b).

This important observation will allow us to get generalizations of Theorem 1.5 to all uncountable regular κ 's (see Theorem 14.10).

Moreover, Remark 12.17(i) still holds after replacing the $\mathcal{L}_{\kappa+\kappa}$ -sentence Ψ with the $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -sentence $\bar{\Psi}$. Remark 12.17(ii) can be improved to the following Remark 13.14(i), and since only bonded formulæ are now involved in it, we further get Remark 13.14(ii)

- REMARKS 13.14. (i) Every model X of $\bar{\Psi}$ (and hence every model of $\bar{\sigma}_T$ for $T \in \mathbb{T}_\kappa$) admits an $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -Scott sentence $\bar{\sigma}^X$. The sentence $\bar{\sigma}^X$ is again obtained by applying the argument of Corollary 13.9 (see also Corollary 12.12) to the singleton of $\bar{h}_T(X)$.
- (ii) By the previous observation and Corollary 8.10(b), it follows that if κ is regular then the isomorphism type $\{Y \in \text{Mod}_{\bar{\mathcal{L}}}^\kappa \mid Y \cong X\}$ of every $X \in \text{Mod}_{\bar{\mathcal{L}}}^\kappa$ is an effective $\kappa + 1$ -Borel set with respect to the bounded topology.

Notice that if $\kappa^{<\kappa} = \kappa$, then by the generalized Lopez-Escobar theorem (i.e. Theorem 8.7) a structure $X \in \text{Mod}_{\bar{\mathcal{L}}}^\kappa$ admits an $\mathcal{L}_{\kappa+\kappa}$ -Scott sentence (for \mathcal{L} and arbitrary finite relational language) if and only if its isomorphism type is $\kappa + 1$ -Borel with respect to the bounded (equivalently, to the product, or to the λ -) topology, i.e. (i) and (ii) are essentially equivalent once we drop the restriction $\bar{\sigma}^X \in \bar{\mathcal{L}}_{\kappa+\kappa}^b$. However, recall that in most of the applications considered in this paper, either we will work in models of AD (where AC fails), or we will deal with cardinals κ which are strictly smaller than 2^{\aleph_0} (see the discussions in Sections 1 and 9), so in both cases we lack the crucial condition $\kappa^{<\kappa} = \kappa$, even when e.g. $\kappa = \omega_1$. This is why we preferred to consider separately the two statements (i) and (ii) of Remark 13.14.

As in the case of Remark 13.14(i) (see the discussion after Remarks 12.17), Remark 13.14(ii) is again well-known when $\kappa = \omega$ (for any countable language \mathcal{L} and any countable structure $X \in \text{Mod}_{\mathcal{L}}^{\omega}$, see e.g. [Kec95, Theorem 16.6]). However, when κ is uncountable then such condition may fail again for some $X \in \text{Mod}_{\mathcal{L}}^{\kappa}$: when $\kappa^{<\kappa} = \kappa$ and AC is assumed this already follows from Example 12.18 and Theorem 8.7, but since as discussed above we will often work in a different context we point out the following independent counterexample (again due to S. D. Friedman).

EXAMPLE 13.15. Let \mathcal{L} be the order language, $\kappa = \omega_1$, and endow both ${}^{\omega_1}2$ and $\text{Mod}_{\mathcal{L}}^{\omega_1}$ with the bounded topology. Consider the structure

$$X := \omega_1 \times \mathbb{Q}_1,$$

where \mathbb{Q}_1 is defined as in Example 12.18 and X is endowed with the lexicographical ordering: we claim that the isomorphism type of X is not $\kappa + 1$ -Borel. Otherwise, it would be $\kappa + 1$ -Borel also the club filter $\text{CUB} \subseteq {}^{\omega_1}2$ defined by

$$\text{CUB} := \{x \in {}^{\omega_1}2 \mid \{\alpha < \omega_1 \mid x(\alpha) = 1\} \text{ contains a club of } \omega_1\}$$

(because there is a continuous map $f: {}^{\omega_1}2 \rightarrow \text{Mod}_{\mathcal{L}}^{\omega_1}$ such that $f^{-1}(\{Y \in \text{Mod}_{\mathcal{L}}^{\omega_1} \mid Y \cong X\}) = \text{CUB}$, see below). But this is impossible, because every $\kappa + 1$ -Borel subset $B \subseteq {}^{\omega_1}2$ has the κ -Baire property by Proposition 6.14, while CUB does not have the κ -Baire property (see [HS01]): in fact, for every intersection $C \subseteq {}^{\omega_1}2$ of at most ω_1 -many open dense subsets of ${}^{\omega_1}2$ and every $s \in {}^{<\omega_1}2$ there are $x, y \in C$ such that $x \in \mathbf{N}_s \setminus \text{CUB}$ (which implies that $U \triangle \text{CUB}$ is non-meager for every nonempty open set U) and $y \in \text{CUB}$ (which implies that CUB itself is non-meager), so Proposition 6.15 gives the desired result. The reduction f of CUB to the isomorphism class of X is defined as follows: given $x \in {}^{\omega_1}2$, let $f(x)$ be (an isomorphic copy on ω_1 of) the structure Y_x obtained by ordering lexicographically the set³⁶

$$\{0\} \times \mathbb{Q}_1 \cup (\{\alpha \mid x(\alpha) = 1\} \times \mathbb{Q}_1) \cup (\{\alpha \mid x(\alpha) = 0\} \times \mathbb{Q}_0),$$

where $\mathbb{Q}_0, \mathbb{Q}_1$ are again defined as in Example 12.18 (it is left to the reader to verify that we can identify each Y_x with an isomorphic copy on ω_1 in such a way that the resulting f is continuous). Notice that, in particular, when $x \in {}^{\omega_1}2$ is constantly equal to 1 then $Y_x = X$. If $x \in \text{CUB}$, let $C \subseteq \omega_1$ be a club of ω_1 such that $0 \in C$ and $x(\alpha) = 1$ for every $0 \neq \alpha \in C$, and fix an order preserving bijection $j: C \rightarrow \omega_1$. Then the map i sending $(\alpha, 0)$ to $(j(\alpha), 0)$ (for $\alpha \in C$) can be extended to an isomorphism between Y_x and X (using the fact that any two countable dense linear orders without maximum and minimum are isomorphic to \mathbb{Q} , and hence to each other). Conversely, if $Y_x \cong X$ then Y_x contains a closed unbounded (with respect to its ordering) set of order type ω_1 , whence $x \in \text{CUB}$. Therefore $x \in \text{CUB} \Leftrightarrow f(x) \cong X$, as required.

14. Definable cardinality and reducibility

As recalled in Section 2.3, under AC the cardinality $|X|$ is defined to be the unique cardinal κ in bijection with X . The resulting theory, though, is somewhat unsatisfactory, since there are no universally accepted methods to settle simple questions like e.g. the continuum problem: by work of Gödel [Göd38] and Cohen [Coh63] the theory ZFC does not settle the statement $2^{\aleph_0} = \aleph_1$. Moreover, knowing that $|\mathbb{R}| = \kappa$ for some cardinal κ , gives little information on such κ or on the possible bijections between \mathbb{R} and κ : in fact, there is no *definable* or *natural* way to explicitly well-order the reals in ZFC (see e.g. [SW90]). This should be contrasted with the fact that, in practice, when $|I| \leq |J|$ one would like to have an explicit witness for this fact, namely a reasonably defined injection from I to J . These observations yield the notion of *definable*

³⁶We define the initial segment $\{0\} \times \mathbb{Q}_1$ of Y_x independently from x because we want to guarantee that Y_x always has a minimal element, as X does.

cardinality. Several definitions of this concept have been considered in the literature (usually depending on the kind of problem one is dealing with), and they all amount to restricting the objects under consideration and the functions used to compute cardinalities to some reasonably simple class, e.g. to the class of functions belonging to some inner model such as $L(\mathbb{R})$, $OD(\mathbb{R})$, and so on. It is quite remarkable that when replacing the notion of cardinality with its definable version, all obstacles (i.e. the independence results on the size of simple sets) simply disappear (see Section 14.3).

As for cardinalities, also the notion of reducibility is not completely satisfactory unless we impose some sort of definability condition on (i.e. we restrict the class of) the reductions that can be used. For example, let us consider the Vitali equivalence relation E_0 on the real line \mathbb{R} defined by

$$(14.1) \quad r E_0 r' \Leftrightarrow r - r' \in \mathbb{Q}.$$

Under AC we have that $E_0 \leq \text{id}(\mathbb{R})$, but on the other hand there is no Baire-measurable (in particular, no Borel) witness for this reduction. This kind of phenomenon suggests that it could be more interesting to consider what could be called *definable reducibility*. Again, this is a vague notion and we will need to specify which definability conditions we are interested in.

By the observations contained in Section 2.6.4, the notions of (definable) cardinality and of (definable) reducibility are strictly related, and all results about (definable) reducibility can be immediately translated into results about (definable) cardinality. For example, we can e.g. derive from the subsequent Theorem 15.16 and Remark 15.17 a nice characterization of small cardinalities (see Section 2.6.4) in terms of infinitary sentences, or, to be more precise, in terms of the bi-embeddability relation on the models of such sentences: in fact, under $\text{AD}_{\mathbb{R}}$ the cardinality of any set A is small if and only if there is a (regular) cardinal $\kappa < \Theta$ and an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ such that $|A| = |\text{Mod}_{\sigma}^{\kappa} / \approx|$, where \approx denotes as usual the bi-embeddability relation. In other words, under $\text{AD}_{\mathbb{R}}$ the set of small cardinalities can be written as

$$\{|\text{Mod}_{\sigma}^{\kappa} / \approx| \mid \kappa < \Theta, \sigma \in \mathcal{L}_{\kappa+\kappa}\}.$$

In this section we will consider many different incarnations of definable cardinality and definable reducibility which have appeared in the literature, and using these notions (or naturally adapting them to our new setup) we will show that the completeness and invariant universality results obtained in Sections 11–13 can be interpreted as results on definable reducibility and hence, by the discussion above, also on definable cardinality.

14.1. Topological complexity. The first case of definable reducibility we will analyze is the Borel reducibility $\leq_{\mathbf{B}}$: as discussed in the introduction, in this case the objects are *analytic* relations on ${}^{\omega}2$ or, equivalently, on any uncountable Polish or standard Borel space, and the reductions are *Borel functions* between such spaces. This is nowadays the standard reducibility notion between Σ_1^1 quasi-orders and equivalence relations — see e.g. [BK96, Hjo00, KM04, Kan08, Gao09]. We will generalize this notion to our context by considering $\kappa + 1$ -Borel functions (instead of just Borel functions) as reductions between quasi-orders defined on (subsets of) spaces of type λ , with λ an infinite cardinal. *From this point onward, unless otherwise explicitly stated, we will conform to the standard practice in the existing literature of endowing all such spaces with the bounded topology τ_b , and hence continuity and $\kappa + 1$ -Borelness (of both functions and sets) will always tacitly refer to this topology.*

When replacing Borel functions with $\kappa + 1$ -Borel functions, we have to decide whether we want to use the strong or the weak formulation of such class of functions (see Definition 5.1). Of course the smaller is the class of reducing functions, the stronger are the results asserting that a certain quasi-order is reducible to another: so in general we should prefer $\kappa + 1$ -Borel reductions to the weakly $\kappa + 1$ -Borel ones. However, as discussed after Proposition 5.2, when

considering functions $f: {}^\lambda 2 \rightarrow {}^\mu 2$ with $\lambda < \mu$ it is often more meaningful to consider the weaker notion of $\kappa + 1$ -Borelness — even the inclusion map f defined in Proposition 5.2(b) can fail to be (non-weakly) $\kappa + 1$ -Borel. The next example shows that a similar phenomenon appears when considering reducibilities between quasi-orders.

EXAMPLE 14.1. Let κ and $\lambda < \mu$ be infinite cardinals, and assume that $\mathbf{B}_{\kappa+1}({}^\lambda 2, \tau_b) \neq \mathcal{P}({}^\lambda 2)$. Let E be the identity relation on ${}^\lambda 2$ (i.e. $x E y \Leftrightarrow x = y$), and let F be its “extension” to the space ${}^\mu 2$, that is set for $x, y \in {}^\mu 2$

$$x F y \Leftrightarrow x \upharpoonright \lambda = y \upharpoonright \lambda.$$

The map ${}^\mu 2 \rightarrow {}^\lambda 2$ sending $x \in {}^\mu 2$ to $x \upharpoonright \lambda$ is continuous (and hence also $\kappa + 1$ -Borel), and reduces F to E . However, there is no $\kappa + 1$ -Borel reduction of E to F . Indeed, let $f: {}^\lambda 2 \rightarrow {}^\mu 2$ be a reduction of E to F , $A \subseteq {}^\lambda 2$ be a set not in $\mathbf{B}_{\kappa+1}({}^\lambda 2, \tau_b)$, and set

$$A' := \{f(x) \upharpoonright \lambda \mid x \in A\}.$$

Then $U := \bigcup_{s \in A'} N_s^\mu$ is τ_b -open, but $f^{-1}(U) = A$ since f reduces E to F , whence $f^{-1}(U) \notin \mathbf{B}_{\kappa+1}({}^\lambda 2, \tau_b)$.

Example 14.1 shows that if we decide to consider only (non-weakly) $\kappa + 1$ -Borel functions as reductions, then the identity relation E on ${}^\lambda 2$ would be *strictly* more complicated than its “extension” F to ${}^\mu 2$: this contradicts our intuition, which suggests that E and F should have the same complexity with respect to any reasonable notion of reducibility. All this discussion justifies the forthcoming definition of $\kappa + 1$ -Borel reducibility, and our choice of the reducing functions should then appear quite natural.

DEFINITION 14.2. Let λ, μ, κ be infinite cardinals, and let \mathcal{X}, \mathcal{Y} be spaces of type λ and μ , respectively. Let R and S be quasi-orders defined on (subsets of) \mathcal{X} and \mathcal{Y} , respectively. We say that R is $\kappa + 1$ -**Borel reducible** to S (in symbols $R \leq_{\mathbf{B}}^\kappa S$) if and only if:³⁷

case $\lambda \geq \mu$: there is a $\kappa + 1$ -Borel reduction $f: \text{dom}(R) \rightarrow \text{dom}(S)$ of R to S ;

case $\lambda < \mu$: there is a *weakly* $\kappa + 1$ -Borel function $f: \text{dom}(R) \rightarrow \text{dom}(S)$ reducing R to S .

The quasi-orders R and S are $\kappa + 1$ -**Borel bi-reducible** (in symbols $R \sim_{\mathbf{B}}^\kappa S$) if both $R \leq_{\mathbf{B}}^\kappa S$ and $S \leq_{\mathbf{B}}^\kappa R$.

Finally, when replacing (weakly) $\kappa + 1$ -Borel functions with their effective counterparts in the previous definitions, we get the notions of **effective** $\kappa + 1$ -**Borel (bi-)reducibility**, denoted by $\leq_{\mathbf{B}}^\kappa$ and $\sim_{\mathbf{B}}^\kappa$, respectively.

Notice that when $\lambda = \mu = \kappa = \omega$, the quasi-order $\leq_{\mathbf{B}}^\kappa$ coincide with the usual Borel reducibility $\leq_{\mathbf{B}}$ mentioned in the introduction and at the beginning of this subsection. Therefore the next completeness result naturally generalizes Theorem 1.1 to uncountable κ 's.

THEOREM 14.3. *Let κ be an infinite cardinal and R be a κ -Souslin quasi-order on ${}^\omega 2$. Then $R \leq_{\mathbf{B}}^\kappa \sqsubset_{\text{CT}}^\kappa$. In particular, $\sqsubset_{\text{CT}}^\kappa$ is $\leq_{\mathbf{B}}^\kappa$ -complete for the class of κ -Souslin quasi-orders on ${}^\omega 2$, and the same is true when replacing $\leq_{\mathbf{B}}^\kappa$ with $\leq_{\mathbf{B}}$.*

PROOF. The function $f_T: {}^\omega 2 \rightarrow \text{CT}_\kappa$ defined in (11.5) is easily seen to be continuous when both spaces are endowed with the *product* topology τ_p . Since the topologies τ_p and τ_b coincide on ${}^\omega 2$, this shows that f_T is (effective) weakly $\kappa + 1$ -Borel (indeed, it is even effective weakly α -Borel for any $\alpha \geq \omega$). Since by Theorem 11.8(a) the function f_T reduces R to $\sqsubset_{\text{CT}}^\kappa$, the result follows. The additional part for $\kappa = \lambda^+$ a successor cardinal follows from the same argument. \square

³⁷See Remark 7.10 for more on *partial* (weakly) $\kappa + 1$ -Borel functions.

As observed in Section 5.1.2, there is another natural generalization of the notion of Borel functions, namely Γ -in-the-codes functions (for suitable boldface pointclasses Γ): this yields a corresponding generalization of the Borel reducibility \leq_B .

DEFINITION 14.4. Let Γ be a boldface pointclass, κ be an infinite cardinal for which there is a Γ -code (so that $\kappa \leq \delta_\Gamma$ by Remark 5.4(ii)), and \mathcal{X} be a space of type κ . Given two quasi-orders R and S on, respectively, ${}^\omega 2$ and (a subset of) \mathcal{X} , we say that R is **Γ -reducible** to S (in symbols, $R \leq_\Gamma S$) if there is a Γ -in-the-codes function $f: {}^\omega 2 \rightarrow \text{dom}(S) \subseteq \mathcal{X}$ which reduces R to S .

REMARK 14.5. Since one can speak of Γ -in-the-codes functions only if there is a Γ -code for κ , in all the results mentioning the reducibility \leq_Γ we will tacitly assume that such a code exists. When assuming AD this is actually granted by Proposition 9.25(b). Similarly, in the AC world this implicit assumption will in most cases follow from the other hypotheses (see e.g. Theorem 15.8 and the ensuing remark), although there are some cases, such as Theorem 15.6, in which the situation is less clear.

Thus, in particular, $\leq_{\Sigma_1^1}$ coincides with \leq_B (notice that $\Gamma = \Sigma_1^1$ automatically implies $\kappa = \omega$ in Definition 14.4, so that in this case \mathcal{X} is homeomorphic to the classical Cantor space ${}^\omega 2$). The next theorem can thus be seen as another natural generalization to uncountable κ 's of Theorem 1.1.

- THEOREM 14.6. (a) (AC) Let $\kappa \leq 2^{\aleph_0}$ be such that there is a $\mathcal{S}(\kappa)$ -code for it. Then the embeddability relation \sqsubseteq_{CT}^κ is $\leq_{\mathcal{S}(\kappa)}$ -complete for the class of κ -Souslin quasi-orders on ${}^\omega 2$, i.e. $R \leq_{\mathcal{S}(\kappa)} \sqsubseteq_{CT}^\kappa$ for every κ -Souslin quasi-order R on ${}^\omega 2$.
 (b) (AD + DC) Let κ be a Souslin cardinal. Then the embeddability relation \sqsubseteq_{CT}^κ is $\leq_{\mathcal{S}(\kappa)}$ -complete for the class of κ -Souslin quasi-orders on ${}^\omega 2$.

Observe that in Theorem 14.6 it always makes sense to consider the notion of $\leq_{\mathcal{S}(\kappa)}$ -reducibility (for part (b) use Proposition 9.25(b)). Moreover, recall that the condition on κ in part (a) is automatically satisfied if $\kappa = \omega_1$, $\kappa = \omega_2$, or $\kappa = 2^{\aleph_0}$ by Proposition 9.16(b).

PROOF. Let $R = p[T]$ be a κ -Souslin quasi-order on ${}^\omega 2$. As noticed at the beginning of the proof of Theorem 14.3, the function $f_T: {}^\omega 2 \rightarrow CT_\kappa \subseteq \text{Mod}_\kappa^\kappa$ defined in (11.5) is easily seen to be continuous when both spaces are endowed with the product topology τ_p , so for every U in the canonical basis $\mathcal{B}_p(\text{Mod}_\kappa^\kappa)$ for the product topology we have

$$f_T^{-1}(U) \in \Sigma_1^0({}^\omega 2) \subseteq \Delta_1^1({}^\omega 2) \subseteq \Delta_{\mathcal{S}(\kappa)}({}^\omega 2).$$

By Propositions 9.19 and 9.28, under our assumptions f_T is then $\mathcal{S}(\kappa)$ -in-the-codes. Since f_T reduces R to \sqsubseteq_{CT}^κ by Theorem 11.8(a), we get $R \leq_{\mathcal{S}(\kappa)} \sqsubseteq_{CT}^\kappa$, as required. \square

We now consider some results concerning invariant universality which generalize Theorems 1.4 and 1.5 to regular uncountable κ 's. Given an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ , say that a map $h: \text{Mod}_\sigma^\kappa \rightarrow {}^\omega 2$ is **$\mathcal{L}_{\kappa+\kappa}$ -measurable** if for every open set $U \subseteq {}^\omega 2$ (equivalently, for every basic open set $U = \mathbf{N}_s^\omega$ with $s \in {}^{<\omega} 2$) there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ_U such that $h^{-1}(U) = \text{Mod}_{\sigma_U}^\kappa \cap \text{Mod}_\sigma^\kappa = \text{Mod}_{\sigma_U \wedge \sigma}^\kappa$. Similarly, we can define the notion of **$\mathcal{L}_{\kappa+\kappa}^b$ -measurability**.

Notice that by Corollary 8.10(b), if κ is regular and $h: \text{Mod}_\sigma^\kappa \rightarrow {}^\omega 2$ is $\mathcal{L}_{\kappa+\kappa}^b$ -measurable, then h is also (effective) $\kappa + 1$ -Borel measurable.

LEMMA 14.7. The map $h_T: \text{Mod}_{\sigma_T}^\kappa \rightarrow {}^\omega 2$ from Definition 12.13 is $\mathcal{L}_{\kappa+\kappa}^b$ -measurable, and also (effective) $\kappa + 1$ -Borel measurable when κ is regular,.

PROOF. Let f_T be the function defined in (11.5), and Ψ be the $\mathcal{L}_{\kappa+\kappa}$ -sentence defined in (12.3). Then for every open set $U \subseteq {}^\omega 2$

$$h_T^{-1}(U) = \{X \in \text{Mod}_\Psi^\kappa \mid \exists x \in U (X \cong f_T(x))\} = g^{-1}((g \circ f_T)(U)).$$

Since $C := {}^\omega 2 \setminus U$ is closed, $(g \circ f_T)(C)$ is closed as well by Corollary 12.11, so let σ_V be the $\mathcal{L}_{\kappa+\kappa}^b$ -sentence given by Lemma 12.9 applied to $V := {}^{<\omega 2 \times \kappa} 2 \setminus (g \circ f_T)(C)$. Then $h_T^{-1}(U) = g^{-1}((g \circ f_T)(U)) = g^{-1}(V) \cap \text{Mod}_{\sigma_T}^\kappa = \text{Mod}_{\sigma_V}^\kappa \cap \text{Mod}_{\sigma_T}^\kappa$. \square

The next theorem generalizes Theorem 1.4 to *regular* uncountable κ 's.

THEOREM 14.8. *Let κ be an infinite regular cardinal. Then $\sqsubseteq_{\text{CT}}^\kappa$ is $\leq_{\mathbf{B}}^\kappa$ -invariantly universal for κ -Souslin quasi-orders on ${}^\omega 2$, that is: for every κ -Souslin quasi-order R on ${}^\omega 2$ there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ all of whose models are combinatorial trees such that $R \sim_{\mathbf{B}}^\kappa \sqsubseteq_\sigma^\kappa$, and in fact even $R \sim_{\mathbf{B}}^\kappa \sqsubseteq_\sigma^\kappa$.*

PROOF. Let $T \in \mathbb{T}_\kappa$ be such that $R = p[T]$, let σ_T , f_T and h_T be as in Theorem 12.15, and set $\sigma := \sigma_T$. Since by Theorem 12.15 we already know that f_T and h_T reduce R and $\sqsubseteq_\sigma^\kappa$ to each other, it remains to show that they are (weakly) $\kappa + 1$ -Borel functions: arguing as in the proof of Theorem 14.3, we get that the function f_T is (effective) weakly $\kappa + 1$ -Borel (and in fact also effective weakly α -Borel for any $\alpha \geq \omega$), while the function h_T is (effective) $\kappa + 1$ -Borel by regularity of κ and Lemma 14.7. \square

Notice that the $\mathcal{L}_{\kappa+\kappa}$ -sentence σ we provided in the proof of Theorem 14.8 (which is the $\mathcal{L}_{\kappa+\kappa}$ -sentence σ_T from Theorem 12.15, see also Corollary 12.12) does not belong to the fragment $\mathcal{L}_{\kappa+\kappa}^b$,³⁸ and hence we cannot in general guarantee that Mod_σ^κ be a $\kappa + 1$ -Borel subset of $\text{Mod}_\mathcal{L}^\kappa$ (see Section 8.2, and in particular Remark 8.12). Therefore, if one aims at generalizing Theorem 1.5 to (regular) uncountable κ 's, then Theorem 12.15 needs to be replaced with Theorem 13.12 (and CT_κ with OCT_κ). First notice that by replacing in the proof of Lemma 14.7 the map f_T with the function \bar{f}_T from (13.2), Ψ with the $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -sentence $\bar{\Psi}$ from (13.4), Corollary 12.11 with Corollary 13.8, and Lemma 12.9 with Lemma 13.6, we get the following variant of it.

LEMMA 14.9. *The map $\bar{h}_T: \text{Mod}_{\sigma_T}^\kappa \rightarrow {}^\omega 2$ from Definition 13.10 is $\bar{\mathcal{L}}_{\kappa+\kappa}^b$ -measurable, and hence, if κ is regular, also (effective) $\kappa + 1$ -Borel measurable.*

Then arguing as in the proof of Theorem 14.8 we get the following generalization of Theorem 1.5.

THEOREM 14.10. *Let κ be an infinite regular cardinal. Then for every κ -Souslin quasi-order R on ${}^\omega 2$ there is an effective $\kappa + 1$ -Borel set $B \subseteq \text{Mod}_\mathcal{L}^\kappa$ closed under isomorphism (all of whose elements are ordered combinatorial trees) such that $R \sim_{\mathbf{B}}^\kappa \sqsubseteq \restriction B$, and in fact even $R \sim_{\mathbf{B}}^\kappa \sqsubseteq \restriction B$.*

PROOF. Let $T \in \mathbb{T}_\kappa$ be such that $R = p[T]$, let $\bar{\sigma}_T$, \bar{f}_T and \bar{h}_T be as in Theorem 13.12, and set $\bar{\sigma} := \bar{\sigma}_T$. Then $B := \text{Mod}_{\bar{\sigma}_T}^\kappa \subseteq \text{Mod}_\mathcal{L}^\kappa$ is effective $\kappa + 1$ -Borel by (13.9). Since by Theorem 13.12 we know that \bar{f}_T and \bar{h}_T reduce R and $\sqsubseteq_{\bar{\sigma}}^\kappa$ to each other, it remains to show that they are both (weakly) $\kappa + 1$ -Borel functions. For the function \bar{f}_T we argue as in the proof of Theorem 14.3: $\bar{f}_T: {}^\omega 2 \rightarrow \text{Mod}_\mathcal{L}^\kappa$ is easily seen to be continuous when both spaces are endowed with the product topology τ_p , and hence it is (effective) weakly $\kappa + 1$ -Borel (in fact, even effective weakly α -Borel for any $\alpha \geq \omega$). The function \bar{h}_T is effective $\kappa + 1$ -Borel by regularity of κ and Lemma 14.9, so we are done. \square

³⁸More precisely: the sentence σ_T provided in the proof of Corollary 12.12 is the conjunction of an $\mathcal{L}_{\kappa+\kappa}^b$ -sentence with the $\Psi \in \mathcal{L}_{\kappa+\kappa}$ from (12.3). Since the latter does not belong to the bounded logic $\mathcal{L}_{\kappa+\kappa}^b$, the same applies to σ_T .

14.2. Absolutely definable reducibilities. Even if Borel reducibility (i.e. Σ_1^1 -reducibility) is probably the most natural way to compare the complexity of Σ_1^1 equivalence relations and quasi-orders, one sometimes needs to generalize this notion to the projective levels: for example, in [HK95, Hjo00] the so-called *absolutely Δ_2^1 -reducibility* $\leq_{a\Delta_2^1}$, i.e. the reducibility notion obtained using absolutely Δ_2^1 maps as reductions, has proven to be useful in this area. Here we recall the definition of such functions as presented in [Hjo00].

DEFINITION 14.11 ([Hjo00, Definition 9.1]). Let HC be the collection of hereditarily countable sets. A function $f: {}^\omega\omega \rightarrow \text{HC}$ is *absolutely Δ_2^1* if there is some parameter $p \in {}^\omega\omega$ and an LST-formula $\Psi(x_0, x_1, z_0)$ such that:

- (1) for all $x \in {}^\omega\omega$ and $y \in \text{HC}$, $f(x) = y$ if and only if there is some countable transitive set \mathcal{M} including x , y , and p such that $(\mathcal{M}, \in) \models \Psi[x, y, p]$;
- (2) $\Psi(x_0, x_1, p)$ *absolutely* defines a function, in the sense that in all generic extensions of the universe V it continues to be the case that for all $x \in {}^\omega\omega$ there exists $y \in \text{HC}$ and a countable transitive \mathcal{M} with $(\mathcal{M}, \in) \models \Psi[x, y, p]$.

Such definition is then naturally extended to cover all functions of interest in [Hjo00, Section 9], including functions between arbitrary Polish spaces, functions between ω_1 and HC, and so on.

REMARK 14.12. The reference to the countable transitive \mathcal{M} in Definition 14.11 is added to have a Δ_2^1 definition of f , but if we are only interested in the *absolute definability* of f (without specifying the complexity of such a definition) then it is natural to just ask that the LST-formula $\Psi(x_0, x_1, p)$ defines (in V) the graph of $f: {}^\omega\omega \rightarrow \text{HC}$ and continues to define a function between ${}^\omega\omega$ and HC in all generic extensions of the universe V .

In this section we will generalize this approach and show that, essentially, the reductions obtained in the proof of Theorem 12.15 cannot be destroyed by passing to set-forcing extensions or to inner models. Since these are the two main techniques for proving the independence of a given mathematical assertion from the chosen set-theoretic axiomatization, the subsequent Theorem 14.19 essentially shows that the invariant universality of the embeddability relation \sqsubset_{CT}^κ is absolute (in a very strong sense specified below) for transitive models of ZF containing all relevant parameters.

REMARK 14.13. In the classical context, $(\omega + 1)$ -Borel functions between Polish spaces are absolutely Δ_2^1 in the sense of Definition 14.11 by Shoenfield Σ_2^1 -absoluteness. However, when κ is uncountable the notion of a $\kappa + 1$ -Borel set is no more absolute, and therefore the absoluteness of the reductions f_T, h_T considered in Theorem 12.15 does not follow from the results in Section 14.1.

Before stating the main result of this section (Theorem 14.19), we first need to adapt Definition 14.11 to our context. To simplify the presentation, we will give a second-order definition of *absolutely definable* functions (but we will later discuss how such a definition can be formulated in first-order logic, see Examples 14.17). This second-order formulation has the advantage of allowing us to consider not only absoluteness with respect to generic extensions of the universe, as done in Hjorth's Definition 14.11, but also e.g. with respect to inner models as well as to many other set-theoretic constructions which are used to produce suitable transitive models of ZF. One of the most general classes of ZF-models that we can deal with is delimited by the following technical condition.

DEFINITION 14.14. Let κ be an infinite cardinal and M be a transitive model of ZF. We say that M κ -**extends** V (respectively, V κ -**extends** M) if $\kappa \in \text{Card}^M$ and $({}^\kappa\kappa)^M \supseteq ({}^\kappa\kappa)^V$ (respectively, $({}^\kappa\kappa)^M \subseteq ({}^\kappa\kappa)^V$).

The model M is called κ -**compatible** (with V) if either M κ -extends V or V κ -extends M .

Notice that if M κ -extends V then we also have $(\omega 2)^M \supseteq (\omega 2)^V$ and $(\text{Mod}_{\mathcal{L}}^\kappa)^M \supseteq (\text{Mod}_{\mathcal{L}}^\kappa)^V$, while if V κ -extends M then $(\omega 2)^M \subseteq (\omega 2)^V$ and $(\text{Mod}_{\mathcal{L}}^\kappa)^M \subseteq (\text{Mod}_{\mathcal{L}}^\kappa)^V$. Every inner model is κ -compatible (for every cardinal κ of V), and every generic extension of V is κ -compatible for all cardinals κ which are not collapsed by the forcing under consideration.

As done by Hjorth in Definition 14.11, also the following Definition 14.15 is given just for the specific functions which are relevant to the results of this paper. However, with some extra work such a definition could be easily adapted to a general definition of an absolutely definable function f between (definable subsets) of spaces \mathcal{X}, \mathcal{Y} of type, respectively, $\lambda, \mu \in \text{Card}$.

DEFINITION 14.15. Let κ be an infinite cardinal and $\sigma \in \mathcal{L}_{\kappa+\kappa}$. A function $f: \omega 2 \rightarrow \text{Mod}_\sigma^\kappa$ (respectively, $f: \text{Mod}_\sigma^\kappa \rightarrow \omega 2$) is called **absolutely (p -)definable** if there is a parameter $p \in {}^\kappa\kappa$ and an LST-formula $\Psi_f(x_0, x_1, z_0, z_1)$ such that for all κ -compatible transitive models M of ZF such that³⁹ $\sigma, p \in M$, the formula $\Psi_f(x_0, y_0, \kappa, p)$ defines in M (the graph of) a function $f^M: (\omega 2)^M \rightarrow (\text{Mod}_\sigma^\kappa)^M$ (respectively, $f^M: (\text{Mod}_\sigma^\kappa)^M \rightarrow (\omega 2)^M$) such that

$$(14.2) \quad f(x) = f^M(x) \quad \text{for every } x \in \text{dom}(f) \cap \text{dom}(f^M).$$

If p and $\Psi_f(x_0, x_1, z_0, z_1)$ are as above, then we say that $\Psi_f(x_0, x_1, \kappa, p)$ **absolutely defines** the function f .

REMARKS 14.16. (i) Notice that taking $M = V$ in Definition 14.15, we get a condition analogous to Definition 14.11(1), that is: $\Psi_f(x_0, x_1, \kappa, p)$ defines in V the function $f: \omega 2 \rightarrow \text{Mod}_\sigma^\kappa$, i.e. for every $x \in \omega 2$ and $X \in \text{Mod}_\sigma^\kappa$

$$f(x) = X \Leftrightarrow V \models \Psi_f[x, X, \kappa, p]$$

(and similarly for an absolutely definable $f: \text{Mod}_\sigma^\kappa \rightarrow \omega 2$).

- (ii) The coherence condition (14.2) does not explicitly appear in Hjorth's Definition 14.11(2), but it is a very natural requirement (which is satisfied by all reduction considered e.g. in [Hjo00]).
- (iii) Let $f: \omega 2 \rightarrow \text{Mod}_\sigma^\kappa$ be an absolutely definable function and M be as in Definition 14.15. If V κ -extends M , then the function $f^M: (\omega 2)^M \rightarrow (\text{Mod}_\sigma^\kappa)^M$ does not depend on the choice of the parameter p and of the LST-formula $\Psi_f(x_0, x_1, z_0, z_1)$ because we must in any case have $f^M = f \upharpoonright (\omega 2)^M$ by (14.2). In contrast, when M κ -extends V , then the restriction of f^M to $(\omega 2)^M \setminus (\omega 2)^V$ depends in general on the chosen p and Ψ_f (but $f^M \upharpoonright (\omega 2)^V$ does not by (14.2) again). Similar considerations hold for an absolutely definable $f: \text{Mod}_\sigma^\kappa \rightarrow \omega 2$.

As already observed, Definition 14.15 is formally a second-order statement, since the transitive models M of ZF appearing in it are potentially proper classes. However, if we let such M range only over certain kind of ZF-models, then we may turn that definition into a first-order statement (which will in general depend on which type of models M we want to consider).

EXAMPLES 14.17. (A) If we let M vary only on set-forcing extensions of V , as done e.g. in Hjorth's Definition 14.11(2), then Definition 14.15 could be formulated in first-order logic as follows.

A function $f: \omega 2 \rightarrow \text{Mod}_\sigma^\kappa$ is absolutely (p -)definable if there is a parameter $p \in {}^\kappa\kappa$ and an LST-formula $\Psi_f(x_0, x_1, z_0, z_1)$ such that for every forcing notion $\mathbf{P} \in V$ such that $1_{\mathbf{P}} \Vdash \text{"}\kappa \text{ is a cardinal"}$, the following conditions hold:

³⁹Recall from Section 8.1.1 that the statement " $\sigma \in \mathcal{L}_{\kappa+\kappa}$ " is absolute for transitive models of ZF containing κ . This observation will be repeatedly used in the rest of this section without explicitly mentioning it.

- (1) P forces that $\Psi_f(x_0, x_1, \kappa, p)$ defines (the graph of) a function between ${}^\omega 2$ and Mod_σ^κ , that is

$$\begin{aligned} 1_P \Vdash \forall x_0, x_1 (\Psi_f(x_0, x_1, \check{\kappa}, \check{p}) \Rightarrow (x_0 \in \dot{y} \wedge x_1 \in \dot{z})) \\ \wedge \forall x_0 \in \dot{y} \exists! x_1 \in \dot{z} (\Psi_f(x_0, x_1, \check{\kappa}, \check{p})), \end{aligned}$$

where \dot{y} and \dot{z} are the standard P -names such that $\dot{y}^G = ({}^\omega 2)^{V[G]}$ and $\dot{z}^G = (\text{Mod}_\sigma^\kappa)^{V[G]}$, for all generics G ;

- (2) $1_P \Vdash \Psi_f(\check{x}, \check{f}(\check{x}), \check{\kappa}, \check{p})$ for every $x \in {}^\omega 2$.

Notice that since $V \subseteq M$ whenever M is a (set-)generic extension of V , then it is always the case that M κ -extends V and that $\sigma, p \in M$. Therefore we do not need to explicitly mention these conditions in the above reformulation.

- (B) If we are instead interested in inner models $M \subseteq V$ then, using the reflection theorem, we could use the following first-order reformulation of Definition 14.15.

A function $f: {}^\omega 2 \rightarrow \text{Mod}_\sigma^\kappa$ is absolutely (p -)definable if there is a parameter $p \in {}^\kappa \kappa$ and an LST-formula $\Psi_f(x_0, x_1, z_0, z_1)$ such that for every transitive set $M \subseteq V$ such that $\kappa, p, \sigma \in M$ and (M, \in) is a model of (a sufficiently large fragment of) ZF, the following hold:

- (1) $(M, \in) \models \forall x \in {}^\omega 2 \exists! X \in \text{Mod}_\sigma^\kappa \Psi_f[x, X, \kappa, p]$;
- (2) for every $x \in {}^\omega 2 \cap M$ and every $X \in \text{Mod}_\sigma^\kappa \cap M$

$$f(x) = X \Leftrightarrow (M, \in) \models \Psi_f[x, X, \kappa, p].$$

Albeit the first-order formulations of Examples 14.17 would in principle be preferable, we chose the second-order formulation of absolute definability to simplify the presentation in the subsequent proofs and because it allows us to deal at once with all possible interesting situations. Similar observations hold for Definition 14.18 and Theorem 14.19 as well.

Using absolutely definable functions as reductions (in V) between a quasi-order R on ${}^\omega 2$ and an embeddability relation \sqsubset_σ^κ (for some $\mathcal{L}_{\kappa^+ \kappa}$ -sentence σ), we would then have a quite strict analogous of Hjorth's absolutely Δ_1^2 reducibility which may be dubbed *absolutely definable reducibility*. However, when R is a κ -Souslin quasi-order and $T \in \mathbb{T}_\kappa$ is a *faithful representation* of it (see Definition 11.2), then by Remark 11.3 in all transitive κ -compatible ZF-models M containing T we have a canonical extension $R_T^M := (p[T])^M$ of R which is still a κ -Souslin quasi-order and is coherent with R , i.e. R_T^M coincides with R on their common domain $({}^\omega 2)^V \cap ({}^\omega 2)^M$. Therefore when comparing such an R with an embeddability relation \sqsubset_σ^κ , it is very natural to ask that the absolutely definable functions involved continue to be reductions between R_T^M and $(\sqsubset_\sigma^\kappa)^M$ in all ZF-models M as in Definition 14.15 (and not just in V).

DEFINITION 14.18. Let κ be an infinite cardinal, $T \in \mathbb{T}_\kappa$ be a faithful representation of a κ -Souslin quasi-order R , and σ be an $\mathcal{L}_{\kappa^+ \kappa}$ -sentence. We say that R is **absolutely definably reducible** to \sqsubset_σ^κ ,

$$R \leq_{\text{aD}} \sqsubset_\sigma^\kappa$$

if there is a reduction $f: {}^\omega 2 \rightarrow \text{Mod}_\sigma^\kappa$ of R to \sqsubset_σ^κ such that for some parameter $p \in {}^\kappa \kappa$ and some LST-formula $\Psi_f(x_0, x_1, z_0, z_1)$ the following hold:

- (1) $\Psi_f(x_0, x_1, \kappa, p)$ absolutely defines f ;
- (2) for all transitive κ -compatible ZF-models M containing p, T and σ

$$M \models \text{“} f^M \text{ reduces } R_T^M \text{ to } \sqsubset_\sigma^\kappa \text{”},$$

where as in Definition 14.15 f^M is the map defined in M by $\Psi_f(x_0, x_1, \kappa, p)$.

The notions of absolutely definable reducibility of $\sqsubset_{\sigma}^{\kappa}$ to R and of **absolutely definable bi-reducibility** \sim_{aD} are defined similarly.

We are now ready to reformulate our main invariant universality result Theorem 12.15 in terms of absolutely definable reducibility.

THEOREM 14.19. *Let κ be an infinite cardinal. Then $\sqsubset_{\text{CT}}^{\kappa}$ is \leq_{aD} -invariantly universal for κ -Souslin quasi-orders on ${}^{\omega}2$.*

More precisely: for every faithful representation $T \in \mathbb{T}_{\kappa}$ of a κ -Souslin quasi-order R on ${}^{\omega}2$ there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ and two LST-formulae $\Psi_{f_T}(x_0, x_1, z_0, z_1)$ and $\Psi_{h_T}(x_0, x_1, z_0, z_1)$ such that for every transitive κ -compatible ZF-model M with $T \in M$ (and hence $\kappa \in M$) the following conditions hold:

- (a) $\sigma \in (\mathcal{L}_{\kappa+\kappa})^M$ (equivalently, by absoluteness of the statement “ $\sigma \in \mathcal{L}_{\kappa+\kappa}$ ”, $\sigma \in M$);
- (b) $\Psi_{f_T}(x_0, x_1, \kappa, T)$ absolutely defines a function $f_T: {}^{\omega}2 \rightarrow \text{Mod}_{\sigma}^{\kappa}$ witnessing $R \leq_{\text{aD}} \sqsubset_{\sigma}^{\kappa}$;⁴⁰
- (c) $\Psi_{h_T}(x_0, x_1, \kappa, T)$ absolutely defines a function $h_T: \text{Mod}_{\sigma}^{\kappa} \rightarrow {}^{\omega}2$ witnessing $\sqsubset_{\sigma}^{\kappa} \leq_{\text{aD}} R$;
- (d) $h_T^M \circ f_T^M = \text{id}^M$ and $M \models (f_T^M \circ h_T^M)(X) \cong X$ for every $X \in (\text{Mod}_{\sigma}^{\kappa})^M$.

PROOF. Let σ be the $\mathcal{L}_{\kappa+\kappa}$ -sentence obtained in the proof of Corollary 12.12. Let also $\sigma_T^M := (\sigma_T)^M$ be the $(\mathcal{L}_{\kappa+\kappa})^M$ -sentence coming from (the proof of) Corollary 12.12 when applied in M — see Section 12.4. Then

$$(14.3) \quad \sigma = \sigma_T^M$$

by Proposition 12.19, so that $\sigma \in M$ and condition (a) of the theorem holds (this is nontrivial only when $V \kappa$ -extends M).

By Proposition 11.10 we know that there is an LST-formula $\Psi_{f_T}(x_0, x_1, \kappa, T)$ which absolutely defines f_T in the strong sense that: if M is as above, then the function f_T^M defined by $\Psi_{f_T}(x_0, x_1, \kappa, T)$ in M is exactly the function obtained as in (11.5) once all the construction is carried out in M . In particular, since Theorem 11.8(b) holds in M , condition (b) of the theorem holds as well.

Finally, let $\Psi_{h_T}(x_0, x_1, z_0, z_1)$ be the LST-formula from Proposition 12.20(a), and recall that $\Psi_{h_T}(x_0, x_1, \kappa, T)$ defines the function $h_T^M := (h_T)^M$ computed in M according to Definition 12.13, whose domain is $(\text{Mod}_{\sigma_T^M}^{\kappa})^M = (\text{Mod}_{\sigma}^{\kappa})^M$ (the latter equality follows from (14.3)). Since Theorem 12.15 holds in M , both conditions (c) and (d) of the theorem will be satisfied as soon as we will show that condition (14.2) of Definition 14.15 holds for h_T and h_T^M , i.e. that $h_T(X) = h_T^M(X)$ for every $X \in (\text{Mod}_{\sigma}^{\kappa})^V \cap (\text{Mod}_{\sigma}^{\kappa})^M$. Since M is assumed to be κ -compatible, this follows from Proposition 12.20(b), so we are done. \square

REMARKS 14.20. (i) Recall that when $\kappa > \omega$ neither the embeddability relation $\sqsubset \upharpoonright \text{Mod}_{\mathcal{L}}^{\kappa}$ nor the statement “ $X \models \sigma$ ” (for $X \in \text{Mod}_{\mathcal{L}}^{\kappa}$ and σ an arbitrary $\mathcal{L}_{\kappa+\kappa}$ -sentence) are in general absolute for transitive models of ZF (see the observation before Proposition 11.13 and Remark 8.5(iii)). However, for the specific $\mathcal{L}_{\kappa+\kappa}$ -sentences considered in Theorem 14.19 (that is, for those sentences constructed as in the proof of Corollary 12.12) one can show, using Proposition 11.13, that for every transitive κ -compatible ZF-model M , the quasi-orders $(\sqsubset_{\sigma}^{\kappa})^M$ and $(\sqsubset_{\sigma}^{\kappa})^V$ are coherent in the following strong sense:

- (1) if $V \kappa$ -extends M then $(\text{Mod}_{\sigma}^{\kappa})^M = (\text{Mod}_{\sigma}^{\kappa})^V \cap M$, while if $M \kappa$ -extends V then $(\text{Mod}_{\sigma}^{\kappa})^V = (\text{Mod}_{\sigma}^{\kappa})^M \cap V$;

⁴⁰Here T is identified with a parameter in ${}^{\kappa}\kappa$ via its characteristic functions and the bijection $\langle\langle \cdot \rangle\rangle$ from (2.2). The same identification is tacitly applied every time that T is used as a parameter in an LST-formula absolutely defining a function.

- (2) $(\sqsubset_\sigma^\kappa)^M$ and $(\sqsubset_\sigma^\kappa)^V$ coincide on their common domain, that is: for every $X, Y \in (\text{Mod}_\sigma^\kappa)^V \cap (\text{Mod}_\sigma^\kappa)^M$

$$V \models X \sqsubset Y \Leftrightarrow M \models X \sqsubset Y.$$

- (ii) It follows from the absoluteness results above that if e.g. $M \subseteq V$ is a transitive ZF-model containing all the reals of V , and σ_T is an $\mathcal{L}_{\kappa+\kappa}$ -sentence obtained as in Corollary 12.12 starting from some $T \in \mathbb{T}_\kappa \cap M$, then every $X \in \text{Mod}_{\sigma_T}^\kappa$ has an isomorphic copy in M . In other words, $(\text{Mod}_{\sigma_T}^\kappa)^M$ meets all isomorphism classes of $\text{Mod}_{\sigma_T}^\kappa$. Thus $\text{Mod}_{\sigma_T}^\kappa$ is always faithfully represented in every inner model containing $L(\mathbb{R})$.
- (iii) For the sake of simplicity, in this section we just considered absoluteness between the universe of sets V and its generic extensions or inner models. However, our proof shows that the latter constructions can be applied more than once without destroying our absoluteness result (Theorem 14.19). Therefore such a result extends to wider contexts, such as the generic multiverse investigated by several people (including J.D. Hamkins, S.D. Friedman, W.H. Woodin), as long as we move between models of set theory containing the relevant parameters κ and T and in which κ remains a cardinal.

14.3. Reducibilities in an inner model. As a further generalization of absolutely Δ_2^1 -reducibility, one could consider arbitrary reductions in $L(\mathbb{R})$ (between quasi-orders in $L(\mathbb{R})$). This is because the theory of $L(\mathbb{R})$ is canonical under the assumption $\text{AD}^{L(\mathbb{R})}$, and moreover $L(\mathbb{R})$ includes anything one could consider “reasonably” definable, like e.g. all Polish spaces up to homeomorphism, all separable Banach spaces up to linear isometry, all Borel and projective sets and functions, and so forth. This approach to definable reducibility, namely reducibility in $L(\mathbb{R})$, has been explicitly considered e.g. in [Hjo95, Hjo99] and [Hjo00, Chapter 9].

14.3.1. *Cardinality and reducibility in $L(\mathbb{R})$.* Because of the special nature of $L(\mathbb{R})$ (see the subsequent Remark 14.22(i)), many results about $L(\mathbb{R})$ -**reducibility** can be recast in terms of $L(\mathbb{R})$ -**cardinality** and viceversa (see also Section 2.6.4, the introduction to Section 14, and point (D) below).

DEFINITION 14.21. For $A, B \in L(\mathbb{R})$ we let

$$\begin{aligned} |A|_{L(\mathbb{R})} \leq |B|_{L(\mathbb{R})} &\Leftrightarrow \exists f \in L(\mathbb{R}) (f: A \rightarrow B) \\ &\Leftrightarrow L(\mathbb{R}) \models |A| \leq |B|. \end{aligned}$$

As pointed out at the beginning of Section 14, assuming AD the notion of $L(\mathbb{R})$ -cardinality is immune from the phenomenon of independence. The following are a sample of results in this context which are relevant for our discussion. Since according to Definition 14.21, comparing $L(\mathbb{R})$ -cardinalities amounts to work inside $L(\mathbb{R})$, to simplify the notation in our presentation *we will now step into this model and drop the subscripts writing $|A|$ instead of $|A|_{L(\mathbb{R})}$ (and assume AD).*

- (A) Every subset of \mathbb{R} is either countable or else it contains a copy of ${}^\omega 2$ (so there is no intermediate cardinality between ω and $|\mathbb{R}|$), and since \mathbb{R} is not well-orderable, then $|\mathbb{R}|$ is incomparable with (the cardinality of) any uncountable ordinal. This, in a sense, solves the continuum problem under AD.
- (B) More generally, Woodin showed that \mathbb{R} is essentially the unique obstruction for a set A to be well-orderable: either $|A| \leq |\alpha|$ for some ordinal α , or $|\mathbb{R}| \leq |A|$ (see e.g. [Hjo99, Theorem 2.8 and Corollary 2.9]). Note that this generalizes Silver’s dichotomy [Sil80] to arbitrary sets.
- (C) Similarly, Hjorth [Hjo95, Theorem 2.6] extended the Glimm-Effros dichotomy [HKL90] to arbitrary sets A by showing that either $|A| \leq |\mathcal{P}(\alpha)|$ for some ordinal α , or else $|\mathbb{R}/E_0| \leq |A|$, where E_0 is as in (14.1).

- (D) Furthermore, *every* set is of the form $\bigcup_{\alpha < \kappa} A_\alpha$, where $\kappa \in \text{Card}$ and each A_α is of small cardinality, i.e. for every $\alpha < \kappa$ there is an equivalence relation E_α on \mathbb{R} such that $|A_\alpha| = |\mathbb{R}/E_\alpha|$ (see [Hjo95, proof of Theorem 2.6] and [Hjo99, Lemma 2.13]). Therefore all cardinalities in $L(\mathbb{R})$ can be analyzed in terms of ordinals and reducibility between quasi-orders on \mathbb{R} (or, equivalently, on ${}^\omega 2$).

REMARKS 14.22. (i) In (B)–(D) above, it is crucial that we work in $L(\mathbb{R})$, so that, in particular, every set is definable using only reals and ordinals as parameters. However, under AD alone one still has that (B)–(D) are true e.g. for sets of small cardinality, i.e. for sets A such that $|A| = |\mathbb{R}/E|$ for some equivalence relation E on \mathbb{R} (without definability conditions on E), or even in the wider context of real-ordinal definable sets.

- (ii) Dichotomy (B) shows that under AD if A is arbitrary, X is a separable space, and ${}^A X$ is endowed with the product topology,

$${}^A X \text{ is separable} \Leftrightarrow |A| \leq \omega \vee |A| = |\mathbb{R}|.$$

Stepping-back into the universe of sets V (where we may assume that AC holds), all the results above can be restated as assertions about $L(\mathbb{R})$ -cardinalities (as introduced in Definition 14.21) under the assumption $\text{AD}^{L(\mathbb{R})}$. For example (B) reads as follows: assuming $\text{AD}^{L(\mathbb{R})}$, for all $A \in L(\mathbb{R})$ either $|A|_{L(\mathbb{R})} \leq |\alpha|_{L(\mathbb{R})}$ or else $|\mathbb{R}|_{L(\mathbb{R})} \leq |A|_{L(\mathbb{R})}$.

14.3.2. *Cardinality and reducibility in an inner model.* The approach above can be generalized by considering any inner model W of ZF containing all the reals, so that in particular $W \supseteq L(\mathbb{R})$. If W is constructed in a canonical, explicit way, such as $L(\mathbb{R})$ or $\text{OD}(\mathbb{R})$, then it is reasonable to describe the objects in W as *definable*. Assuming e.g. sufficiently large cardinal axioms or determinacy assumptions, or working in some special model of set theory (like the Solovay models), this approach gives in general a nice definable reducibility and cardinality theory. Therefore in what follows we will construe “reasonably definable” as “belonging to the inner model $W \supseteq L(\mathbb{R})$ ” under consideration. The resulting relations of W -**reducibility** \leq_W and W -**bi-reducibility** \sim_W are defined in the obvious way, namely:

DEFINITION 14.23. Let $W \supseteq \mathbb{R}$ be an inner model and $R, S \in W$ be quasi-orders. Then

$$\begin{aligned} R \leq_W S &\Leftrightarrow \exists f \in W \text{ (} f \text{ reduces } R \text{ to } S\text{)} \\ &\Leftrightarrow W \models R \leq S, \end{aligned}$$

and $R \sim_W S \Leftrightarrow R \leq_W S \wedge S \leq_W R$.

Using the fact that they hold in the ZF-model W , Theorems 11.8 and 12.15 can be reformulated as follows:

THEOREM 14.24. Let $W \supseteq \mathbb{R}$ be an inner model and suppose that $\kappa \in \text{Card}^W$ and $R \in \mathbf{S}_W(\kappa)$.⁴¹ Let $S := (\sqsubset_{\text{CT}}^\kappa)^W$ be the embeddability relation between combinatorial trees of size κ in W , i.e. S is the unique quasi-order (in V) such that $W \models “S = \sqsubset_{\text{CT}}^\kappa”$. Then $R \leq_W S$. Therefore S is \leq_W -complete for quasi-orders in $\mathbf{S}_W(\kappa)$.

THEOREM 14.25. Let $W \supseteq \mathbb{R}$ be an inner model and let $\kappa \in \text{Card}^W$. Then for every $R \in \mathbf{S}_W(\kappa)$ there is $\sigma \in (\mathcal{L}_{\kappa^+ \kappa})^W \subseteq \mathcal{L}_{\kappa^+ \kappa}$ such that $R \sim_W \sqsubset_\sigma^\kappa$.

REMARK 14.26. As in [Hjo00, Chapter 9], all terms in the statement of Theorem 14.25 must be relativized to W . Thus e.g. $R \leq_W \sqsubset_\sigma^\kappa$ is construed as: there is $f: \mathbb{R} \rightarrow (\text{Mod}_\sigma^\kappa)^W$ such that for every $x, y \in \mathbb{R}$

$$x R y \Leftrightarrow W \models f(x) \sqsubset_\sigma f(y).$$

⁴¹See Remark 9.6 for the definition of $\mathbf{S}_W(\kappa)$.

Unfortunately, the fact that we are forced to use $(\text{Mod}_\sigma^\kappa)^W$ instead of Mod_σ^κ (and that, in general, $(\text{Mod}_\sigma^\kappa)^W \subset \text{Mod}_\sigma^\kappa$) forbids to formally restate Theorem 14.25 in terms of \leq_W -invariant universality as introduced in Definitions 12.1 and 12.2. Notice also that in (the statement of) Theorem 14.25 we have $(\text{Mod}_\sigma^\kappa)^W = \text{Mod}_\sigma^\kappa \cap W$ by Remark 14.20(i): this is because in this case the formula σ provided by Theorem 12.15 is obtained as in Corollary 12.12, that is it is of the form σ_T for some $T \in (\mathbb{T}_\kappa)^W$. Moreover, by Remark 14.20(ii) we also get that the $\mathcal{L}_{\kappa+\kappa}$ -sentence $\sigma := \sigma_T$ under consideration has the remarkable property that every model (in V) of σ has an isomorphic copy belonging to the inner model W , so that even though they may fail to belong to M , the V -relations \cong_σ^κ and \preceq_σ^κ are at least faithfully represented in W .

The preorder \leq_W from Definition 14.23 can be in principle extended to compare quasi-orders which are not necessarily in W , namely for *arbitrary* quasi-orders R, S of V we can set

$$R \leq'_W S \Leftrightarrow \exists f \in W \text{ (} f \text{ reduces } R \text{ to } S \text{)}.$$

However, by definition $R \leq'_W S$ can hold only if $\text{dom}(R) = \text{dom}(f) \in W$ for some/any f witnessing $R \leq'_W S$. For this reason, we can restate in this more general context only the completeness result (Theorem 11.8) and not the invariant universality one (Theorem 12.15), as in the latter case $\text{Mod}_{\sigma_T}^\kappa$ need not to belong to W — as recalled in Remark 14.26, we are just guaranteed that $\text{Mod}_{\sigma_T}^\kappa \cap W = (\text{Mod}_{\sigma_T}^\kappa)^W \in W$.

THEOREM 14.27. *Let $W \supseteq \mathbb{R}$ be an inner model, κ be a cardinal, and $R \in \mathbf{S}_W(\kappa)$. Then $R \leq'_W \preceq_{\text{CT}}^\kappa$. Therefore $\preceq_{\text{CT}}^\kappa$ is \leq'_W -complete for quasi-orders in $\mathbf{S}_W(\kappa)$.*

PROOF. Let $T \in (\mathbb{T}_\kappa)^W$ be such that $R = p[T]$. Since $\kappa \in \text{Card}^W$ (because W is an inner model) and $T \in W$, by (the proof of) Theorem 14.19 the function f_T from (11.5) is absolutely definable (using only $T \in W$ as a parameter) and reduces R to $\preceq_{\text{CT}}^\kappa$. Since $\mathbb{R} \subseteq W$, by (14.2) we get $f_T^W = f_T$, whence $f_T \in W$. Therefore, f_T witnesses $R \leq'_W \preceq_{\text{CT}}^\kappa$, as required. \square

15. Some applications

As we will see in this section, the results obtained in Sections 11–14 yield several corollaries: inside any model where the κ -Souslin quasi-orders on ${}^\omega 2$ (or, equivalently, on any Polish or standard Borel space) form an interesting class, we get a *completeness* and *invariant universality* result for the embeddability relation on combinatorial trees of size κ . Since there are lot of situations of this kind, in what follows we will just explicitly state a few of them which correspond to some of the most relevant cases. All these results admit several variants which will not be explicitly mentioned but that could be of interest on their own, namely:

- (1) the relations $\leq_{\mathbf{B}}^\kappa$ and $\sim_{\mathbf{B}}^\kappa$ could systematically be replaced by their effective counterparts \leq_B^κ and \sim_B^κ ;
- (2) in each statement, we could equivalently consider the wider class of κ -Souslin quasi-orders defined on *arbitrary Polish spaces* (or even on *arbitrary standard Borel spaces*) instead of its restriction to the κ -Souslin quasi-orders on ${}^\omega 2$ — this is because every two uncountable Polish or standard Borel spaces are Borel isomorphic and $\mathbf{S}(\kappa)$ is closed under Borel (pre)images by Lemma 9.7;
- (3) elementary classes of the form Mod_σ^κ for some $\mathcal{L}_{\kappa+\kappa}$ -sentence σ obtained as in Corollary 12.12 could always be equivalently substituted by Mod_σ^∞ by Remark 12.17(i);
- (4) by Corollary 12.16, in all the subsequent results concerning the bi-reducibility between two quasi-orders R and S we could further add that the quotient orders of R and S are in fact *isomorphic*.

15.1. Σ_2^1 quasi-orders. Recall that since the $\leq_{\Sigma_1^1}$ -reducibility (see Definition 14.4), which is the same as the $\leq_{\mathcal{S}(\omega)}$ -reducibility, coincides with the classical Borel reducibility $\leq_{\mathbf{B}}^{\omega}$, by Theorems 1.1 and 1.4 we have that:

Completeness: $\sqsubset_{\text{CT}}^{\omega}$ is $\leq_{\Sigma_1^1}$ -complete (equivalently, $\leq_{\mathcal{S}(\omega)}$ -complete) for Σ_1^1 quasi-orders on ${}^{\omega}2$, i.e. $R \leq_{\Sigma_1^1} \sqsubset_{\text{CT}}^{\omega}$ (equivalently, $R \leq_{\mathcal{S}(\omega)} \sqsubset_{\text{CT}}^{\omega}$) for every Σ_1^1 quasi-order R on ${}^{\omega}2$;
Invariant universality: $\sqsubset_{\text{CT}}^{\omega}$ is also $\leq_{\mathbf{B}}^{\omega}$ -invariantly universal (and hence also $\leq_{\mathbf{B}}^{\omega}$ -complete) for Σ_1^1 quasi-orders on ${}^{\omega}2$, i.e. for every such R there is an $\mathcal{L}_{\omega_1\omega}$ -sentence σ such that $R \sim_{\mathbf{B}}^{\omega} \sqsubset_{\sigma}^{\omega}$.

The following theorems generalize the above results to the next level of the projective hierarchy. (It is easy to check that in all the situations below it always makes sense to consider Σ_2^1 -in-the-codes and $\mathcal{S}(\omega_1)$ -in-the-codes functions $f: {}^{\omega}2 \rightarrow \text{Mod}_{\mathcal{L}}^{\omega_1}$ by the observations following Proposition 9.18 and Lemma 9.27.)

THEOREM 15.1. *Assume either AC or AD+DC. Then the relation $\sqsubset_{\text{CT}}^{\omega_1}$ is $\leq_{\mathcal{S}(\omega_1)}$ -complete⁴² for Σ_2^1 quasi-orders on ${}^{\omega}2$.*

PROOF. Apply Theorem 14.6, using the fact that $\Sigma_2^1 \subseteq \mathcal{S}(\omega_1)$. \square

Notice that in Theorem 15.1 when assuming AD+DC we could replace the somewhat artificial notion of $\leq_{\mathcal{S}(\omega_1)}$ -completeness with the more natural notion of $\leq_{\Sigma_2^1}$ -completeness because in this case $\mathcal{S}(\omega_1) = \Sigma_2^1$ (see the paragraph before Proposition 9.25). A similar strengthening can be obtained in models of AC as well under some additional set-theoretic assumptions.

THEOREM 15.2 (AC). *Assume either $\text{AD}^{\text{L}(\mathbb{R})}$ or $\text{MA} + \neg\text{CH} + \exists a \in {}^{\omega}\omega (\omega_1^{\text{L}[a]} = \omega_1)$. Then $\sqsubset_{\text{CT}}^{\omega_1}$ is $\leq_{\Sigma_2^1}$ -complete for Σ_2^1 quasi-orders on ${}^{\omega}2$.*

PROOF. Under $\text{ZFC} + \text{MA} + \neg\text{CH} + \exists a \in {}^{\omega}\omega (\omega_1^{\text{L}[a]} = \omega_1)$ we get $\mathcal{S}(\omega_1) = \Sigma_2^1$ by Proposition 9.18, so the result follows from Theorem 15.1.

Let us now assume $\text{AD}^{\text{L}(\mathbb{R})}$, and recall that $\omega_1^{\text{L}(\mathbb{R})} = \omega_1$ and $({}^{\omega}2)^{\text{L}(\mathbb{R})} = {}^{\omega}2$. Let R be a Σ_2^1 quasi-order on ${}^{\omega}2$, so that R is Σ_2^1 also in $\text{L}(\mathbb{R})$ and hence $R \in \mathcal{S}_{\text{L}(\mathbb{R})}(\omega_1)$. Let $T \in (\mathbb{T}_{\omega_1})^{\text{L}(\mathbb{R})}$ be such that $R = p[T]$. Then by (the proof of) Theorem 14.27 the function f_T from (11.5) is a reduction of R to $\sqsubset_{\text{CT}}^{\omega_1}$ and belongs to $\text{L}(\mathbb{R})$. Under our assumptions, $\text{ZF} + \text{AD} + \text{DC}$ holds in $\text{L}(\mathbb{R})$ (whence also $\text{L}(\mathbb{R}) \models \mathcal{S}(\omega_1) = \Sigma_2^1$), so applying Theorem 14.6(b) in $\text{L}(\mathbb{R})$ we have that

$$\text{L}(\mathbb{R}) \models f_T \text{ is } \Sigma_2^1\text{-in-the-codes.}$$

Thus f_T is actually Σ_2^1 -in-the-codes (in V) by Shoenfield's Σ_2^1 -absoluteness. \square

As for invariant universality, we get the following result in $\text{ZF} + \text{AC}_{\omega}(\mathbb{R})$.

THEOREM 15.3 ($\text{AC}_{\omega}(\mathbb{R})$). *The relation $\sqsubset_{\text{CT}}^{\omega_1}$ is $\leq_{\mathbf{B}}^{\omega_1}$ -invariantly universal (and hence also $\leq_{\mathbf{B}}^{\omega_1}$ -complete) for Σ_2^1 quasi-orders on ${}^{\omega}2$.*

PROOF. Since $\Sigma_2^1 \subseteq \mathcal{S}(\omega_1)$, it is enough to use Theorem 14.8, which can be applied because $\text{AC}_{\omega}(\mathbb{R})$ implies that ω_1 is a regular cardinal. \square

Theorems 15.1–15.3 are more interesting in all cases in which the topological notions involved (such as $\mathcal{S}(\omega_1)$ -in-the-codes functions, $\omega_1 + 1$ -Borel sets and functions, and so on) are nontrivial, i.e. when ω_1 is “small enough” with respect to the cardinality of the continuum. This is the case

⁴²By Proposition 9.10, we could replace $\leq_{\mathcal{S}(\omega_1)}$ -completeness with $\leq_{\mathbf{B}}^{\omega_1}$ -completeness. However the latter is a weaker notion, and in fact the corresponding completeness result can be proved already in $\text{ZF} + \text{AC}_{\omega}(\mathbb{R})$ — see Theorem 15.3.

if we work in models of AD. If instead we work in models of ZFC, then as observed in Sections 3–6 and 9 all the relevant topological notions trivialize under CH. Thus we are naturally lead to work in models of ZFC + \neg CH. By the observation following Proposition 9.19, this already gives that the notion of a $\mathcal{S}(\omega_1)$ -in-the-code function $f: {}^\omega 2 \rightarrow \text{Mod}_{\mathcal{L}}^{\omega_1}$ is nontrivial. Moreover, if we further assume that the inequality $2^{\aleph_1} < 2^{(2^{\aleph_0})}$ holds (which is e.g. the case in models of forcing axioms like MA_{ω_1} , PFA, and so on), then also the notions of a (weakly) $\omega_1 + 1$ -Borel function $f: {}^\omega 2 \rightarrow \text{Mod}_{\mathcal{L}}^{\omega_1}$ becomes interesting (see Corollary 5.6(b)). This discussion shows that the following instantiation of Theorems 15.1 and 15.2 is nontrivial.

COROLLARY 15.4 (AC + PFA). *The relation $\sqsubset_{\text{CT}}^{\omega_1}$ is $\leq_{\Sigma_2^1}$ -complete (and hence also $\leq_{\mathcal{S}(\omega_1)}$ - and $\leq_{\mathbf{B}}^{\omega_1}$ -complete) for Σ_2^1 quasi-orders on ${}^\omega 2$.*

PROOF. $\text{AD}^{L(\mathbb{R})}$ follows from PFA by [Ste05], so Theorem 15.2 can be applied. \square

If besides $2^{\aleph_1} < 2^{(2^{\aleph_0})}$ we further assume⁴³ that $2^{(2^{<\aleph_1})} < 2^{(2^{\aleph_1})}$, then also the notion of an $\omega_1 + 1$ -Borel function $f: \text{Mod}_{\mathcal{L}}^{\omega_1} \rightarrow {}^\omega 2$ becomes nontrivial (see Corollary 5.6(c)). Therefore, a situation of interest in which Theorem 15.3 can be applied is the following.

COROLLARY 15.5 (AC). *Assume that $2^\kappa = \kappa^{++}$ for every $\kappa \leq \aleph_3$. Then $\sqsubset_{\text{CT}}^{\omega_1}$ is $\leq_{\mathbf{B}}^{\omega_1}$ -invariantly universal (and hence also $\leq_{\mathbf{B}}^{\omega_1}$ -complete) for Σ_2^1 quasi-orders on ${}^\omega 2$.*

15.2. Projective quasi-orders. In this section we will generalize Theorems 1.1 and 1.4 to even larger projective levels (under suitable assumptions).

15.2.1. Models of AC. Martin showed in [Mar12] that $\text{ZFC} + \forall x \in {}^\omega \omega (x^\# \text{ exists})$ implies that all Σ_3^1 sets are ω_2 -Souslin. Using this fact and applying Theorems 14.6(a) and 14.8, we get the following further generalizations of Theorems 1.1 and 1.4. (Recall that in models with choice it always makes sense to speak of $\mathcal{S}(\omega_2)$ -in-the-codes functions $f: {}^\omega 2 \rightarrow \text{Mod}_{\mathcal{L}}^{\omega_2}$ by the observation following Proposition 9.18, and that every such f is automatically weakly $\omega_2 + 1$ -Borel by Proposition 9.10.)

THEOREM 15.6 (AC). *Assume that $x^\#$ exists for all $x \in {}^\omega \omega$.*

- (a) *The relation $\sqsubset_{\text{CT}}^{\omega_2}$ is $\leq_{\mathcal{S}(\omega_2)}$ -complete for Σ_3^1 quasi-orders on ${}^\omega 2$.*
- (b) *The relation $\sqsubset_{\text{CT}}^{\omega_2}$ is $\leq_{\mathbf{B}}^{\omega_2}$ -invariantly universal (and hence also $\leq_{\mathbf{B}}^{\omega_2}$ -complete) for Σ_3^1 quasi-orders on ${}^\omega 2$.*

An interesting application of this theorem is when considering the quasi-order $(\mathcal{Q}, \leq_{\mathbf{B}})$ of Example 1.6, that is the relation of Borel reducibility between analytic quasi-orders. As recalled in the introduction, such quasi-order may be seen as a (definable) embeddability relation between structures of size the continuum 2^{\aleph_0} : the next result shows that $(\mathcal{Q}, \leq_{\mathbf{B}})$ can be turned into an embeddability relation on structures of size \aleph_2 , *independently of the actual value of 2^{\aleph_0} .*

THEOREM 15.7 (AC). *Assume that $x^\#$ exists for all $x \in {}^\omega \omega$. Then the quotient order of $(\mathcal{Q}, \leq_{\mathbf{B}})$ (definably) embeds into the quotient order of $\sqsubset_{\text{CT}}^{\aleph_2}$.*

Moreover there is an $\mathcal{L}_{\aleph_3 \aleph_2}$ -sentence σ such that the quotient orders of $(\mathcal{Q}, \leq_{\mathbf{B}})$ and $\sqsubset_{\sigma}^{\aleph_2}$ are even (definably) isomorphic.

PROOF. For the additional part about the $\mathcal{L}_{\aleph_3 \aleph_2}$ -sentence σ , we use Corollary 12.16. \square

⁴³Since $2^{\aleph_0} = 2^{\aleph_1}$ implies $2^{(2^{<\aleph_1})} = 2^{(2^{\aleph_1})}$, MA_{ω_1} (and even the stronger PFA, MM, and so on) are not sufficient to ensure the extra cardinal condition under discussion. However, it could still be the case that the notion of an $\omega_1 + 1$ -Borel function $f: \text{Mod}_{\mathcal{L}}^{\omega_1} \rightarrow {}^\omega 2$ is nontrivial also in models of forcing axioms for reasons different from the cardinality considerations of Corollary 5.6.

Recall that the assumption $\forall x \in {}^\omega\omega (x^\# \text{ exists})$ is equivalent over ZFC to Σ_1^1 -determinacy (in fact, even to $<\omega^2$ - Π_1^1 -determinacy) by results of Harrington and Martin (see [Har78]). Assuming more determinacy, we can extend Theorem 15.6 to all projective levels (recall that the axiom $\text{AD}^{\text{L}(\mathbb{R})}$ used in Theorem 15.8 follows both from the existence of infinitely many Woodin cardinals with a measurable above, and from strong forcing axioms such as PFA).

THEOREM 15.8 (AC). *Assume $\text{AD}^{\text{L}(\mathbb{R})}$. Then there is a monotone function $r: \omega \rightarrow \omega$ such that:*

- (a) *the relation $\sqsubset_{\text{CT}}^{\omega_{r(n)}}$ is $\leq_{\mathcal{S}(\omega_{r(n)})}$ -complete for Σ_n^1 quasi-orders on ${}^\omega 2$;*
- (b) *the relation $\sqsubset_{\text{CT}}^{\omega_{r(n)}}$ is $\leq_{\mathbf{B}}^{\omega_{r(n)}}$ -invariantly universal (and hence also $\leq_{\mathbf{B}}^{\omega_{r(n)}}$ -complete) for Σ_n^1 quasi-orders on ${}^\omega 2$.*

An upper bound for r is given by:

$$r(n) \leq \begin{cases} 2^{k+1} - 2 & \text{if } n = 2k + 1, \\ 2^{k+1} - 1 & \text{if } n = 2k + 2. \end{cases}$$

The proof below will in particular show that, under the hypotheses of the theorem, there is an $\mathcal{S}(\omega_{r(n)})$ -code for $\omega_{r(n)}$ for each n , so that in part (a) it makes sense to speak of $\mathcal{S}(\omega_{r(n)})$ -in-the-codes functions.

PROOF. First recall that under our assumptions $\text{L}(\mathbb{R}) \models \text{AD} + \text{DC}$ because in $\text{L}(\mathbb{R})$ the axiom of dependent choice DC follows from AD by [Kec84]. Recall also that projective sets are absolute between V and $\text{L}(\mathbb{R})$, in particular $\delta_n^1 = (\delta_n^1)^{\text{L}(\mathbb{R})}$ (but while δ_n^1 is always a cardinal in $\text{L}(\mathbb{R})$ by $\text{L}(\mathbb{R}) \models \text{AD} + \text{DC}$, it may be just an ordinal in V). Let $\kappa_n := \delta_{n-1}^1 = (\delta_{n-1}^1)^{\text{L}(\mathbb{R})}$ if n is even and $\kappa_n := (\lambda_n^1)^{\text{L}(\mathbb{R})}$ if n is odd, and let $r(n)$ be such that $\omega_{r(n)} = |\kappa_n|$ (where the cardinality of κ_n is computed in V). By Corollary 4.13 we get the above upper bound for the function $r: \omega \rightarrow \omega$ (in the odd case we use the fact that $(\lambda_n^1)^{\text{L}(\mathbb{R})}$ has countable cofinality in $\text{L}(\mathbb{R})$, and hence its V-cardinality is collapsed at least to the V-cardinality of the largest regular cardinal of $\text{L}(\mathbb{R})$ below it). Since $\text{L}(\mathbb{R}) \models \text{AD} + \text{DC}$, by [Jac10, Theorem 2.18] (see the observation before Proposition 9.25) we get

$$\Sigma_n^1 = (\mathcal{S}(\kappa_n))^{\text{L}(\mathbb{R})} \subseteq \mathcal{S}_{\text{L}(\mathbb{R})}(\kappa_n) \subseteq \mathcal{S}(\omega_{r(n)}).$$

Notice that since $(\mathcal{S}(\kappa_n))^{\text{L}(\mathbb{R})} \subseteq \mathcal{S}(\omega_{r(n)})$ and $\kappa_n \geq \omega_{r(n)}$, it follows from Proposition 9.25(b) (applied in the model $\text{L}(\mathbb{R})$ with $\kappa := \kappa_n$) and Remark 5.4(iii) that it always makes sense to speak of $\mathcal{S}(\omega_{r(n)})$ -in-the-codes functions $f: {}^\omega 2 \rightarrow \text{Mod}_{\mathcal{L}}^{\omega_{r(n)}}$ and that, in particular, the hypotheses of Theorems 14.6(a) are satisfied for $\kappa := \omega_{r(n)}$, although it may happen that $\omega_{r(n)} = 2^{\aleph_0}$ if the continuum is smaller than \aleph_ω . Moreover, the $\omega_{r(n)}$ are always successor cardinals, and thus regular in model of AC, so that the hypotheses of Theorem 14.8 are satisfied for $\kappa := \omega_{r(n)}$. Thus, to obtain the desired results it is enough to apply Theorems 14.6(a) and 14.8. \square

15.2.2. Models of AD. By [Jac10, Theorem 2.18], assuming $\text{ZF} + \text{AD} + \text{DC}$ we get that $\Sigma_n^1 = \mathcal{S}(\kappa_n)$, where κ_n is such that $\delta_n^1 = \kappa_n^+$, that is $\kappa_n := \lambda_n^1$ if n is odd and $\kappa_n := \delta_{n-1}^1$ otherwise. Therefore, applying Theorems 14.6(b) and 14.8 we get the following AD-analogue of Theorem 15.8.

THEOREM 15.9 (AD + DC). *Let $0 \neq n \in \omega$.*

- (a) *The relation $\sqsubset_{\text{CT}}^{\kappa_n}$ is $\leq_{\Sigma_n^1}$ -complete for Σ_n^1 quasi-orders on ${}^\omega 2$.*
- (b) *Let n be an even number. Then the relation $\sqsubset_{\text{CT}}^{\delta_{n-1}^1}$ is $\leq_{\mathbf{B}}^{\delta_{n-1}^1}$ -invariantly universal (and hence also $\leq_{\mathbf{B}}^{\delta_{n-1}^1}$ -complete) for Σ_n^1 quasi-orders on ${}^\omega 2$.*

- REMARKS 15.10. (i) In part (a) we can speak of $\leq_{\Sigma_n^1}$ -completeness because in the AD-world $\mathcal{S}(\kappa_n) = \Sigma_n^1$. Notice that $\leq_{\Sigma_n^1}$ -completeness implies $\leq_{\mathbf{B}}^{\kappa_n}$ -completeness by Proposition 9.10, and that the two notions coincide if n is odd by Corollary 9.29.
- (ii) The restriction in part (b) comes from the fact that κ_n is regular if and only if n is even, and therefore Theorem 14.8 can be applied only to the even levels of the projective hierarchy. However, if we drop the requirement that the reductions be (weakly) $\kappa_n + 1$ -Borel, then Theorem 15.9(b) would be true also for the odd levels by Theorem 12.15.

As for Theorem 15.6, also Theorem 15.9 can be applied with $n = 3$ to the quasi-order $(\mathcal{Q}, \leq_{\mathbf{B}})$ of Borel reducibility between analytic quasi-orders: in this case, such definable embeddability is turned into the embeddability relation between structures of size $\kappa_3 = \aleph_\omega$. Since \aleph_ω is a singular cardinal, to get the second part of the following theorem use the observation in Remark 15.10(ii) together with Corollary 12.16, rather than Theorem 15.9(b).

THEOREM 15.11 (AD + DC). *The quotient order of $(\mathcal{Q}, \leq_{\mathbf{B}})$ (definably) embeds into the quotient order of $\sqsubset_{\text{CT}}^{\aleph_\omega}$.*

Moreover there is an $\mathcal{L}_{\aleph_{\omega+1} \aleph_\omega}$ -sentence σ such that the quotient orders of $(\mathcal{Q}, \leq_{\mathbf{B}})$ and $\sqsubset_{\sigma}^{\aleph_\omega}$ are even (definably) isomorphic.

15.3. More complicated quasi-orders in models of determinacy. Assuming ZF + AD + DC, it is natural to consider the largest boldface pointclass to which our results can be applied, namely the collection $\mathcal{S}(\Xi) = \mathcal{S}(\infty)$ of all ∞ -Souslin sets. Assuming AD in $L(\mathbb{R})$, we have that $\Xi = \delta_1^2$ and $\mathcal{S}(\Xi) = \Sigma_1^2$. In this case δ_1^2 is a Souslin cardinal, so applying again Theorems 14.6 and 14.8 we get the following completeness and invariant universality results.

- THEOREM 15.12 (AD + V = $L(\mathbb{R})$). (a) *The embeddability relation $\sqsubset_{\text{CT}}^{\delta_1^2}$ is $\leq_{\Sigma_1^2}$ -complete for Σ_1^2 quasi-orders (equivalently, ∞ -Souslin quasi-orders) on ${}^\omega 2$.*
- (b) *The relation $\sqsubset_{\text{CT}}^{\delta_1^2}$ is $\leq_{\mathbf{B}}^{\delta_1^2}$ -invariantly universal (and hence also $\leq_{\mathbf{B}}^{\delta_1^2}$ -complete) for Σ_1^2 quasi-orders (equivalently, ∞ -Souslin quasi-orders) on ${}^\omega 2$.*

- REMARKS 15.13. (i) In Theorem 15.12 we need not explicitly assume DC as this follows from AD in $L(\mathbb{R})$ by [Kec84].
- (ii) Theorem 14.8 can be applied to get part (b) as Σ_1^2 is also closed under coprojections and therefore the cardinal δ_1^2 is regular by [Mos09, Theorem 7D.8] (in fact, it is a weakly inaccessible cardinal).

Theorem 15.12 can be analogously reformulated in every model of ZF + AD + DC in which Ξ is a Souslin cardinal, i.e. in every model of ZF + AD^+ + DC (see Section 9.4): in fact, in this case Ξ is always a (limit) regular cardinal (see e.g. [Jac08, Lemma 2.20]).

- THEOREM 15.14 (AD^+ + DC). (a) *The embeddability relation $\sqsubset_{\text{CT}}^{\Xi}$ is $\leq_{\mathcal{S}(\Xi)}$ -complete for ∞ -Souslin quasi-orders on ${}^\omega 2$.*
- (b) *The relation $\sqsubset_{\text{CT}}^{\Xi}$ is $\leq_{\mathbf{B}}^{\Xi}$ -invariantly universal (and hence also $\leq_{\mathbf{B}}^{\Xi}$ -complete) for ∞ -Souslin quasi-orders on ${}^\omega 2$.*

REMARK 15.15. Applying Theorem 14.6 and 14.8, a level-by-level version of Theorem 15.14 is obtained. Assuming AD + DC it is possible to assign to every quasi-order R in $\Delta_{\mathcal{S}(\Xi)}$ a regular Souslin cardinal $\kappa_R < \Xi$ such that:

- (a) $R \leq_{\mathcal{S}(\kappa_R)} \sqsubset_{\text{CT}}^{\kappa_R}$;
- (b) $R \sim_{\mathbf{B}}^{\kappa_R} \sqsubset_{\sigma}^{\kappa_R}$ for some $\mathcal{L}_{\kappa_R^+ \kappa_R}$ -sentence σ all of whose models are combinatorial trees.

To see this, it is enough to let $\kappa_R < \Xi$ be the smallest *regular* Souslin cardinal such that $R \in \mathcal{S}(\kappa_R)$, whose existence is granted by Lemma 9.27.

Further assuming $\text{AD}_{\mathbb{R}}$, we get global completeness and invariant universality results. Notice that in this case it does not make much sense to consider $\leq_{\mathcal{S}(\Theta)}$ -reducibility, since under $\text{ZF} + \text{AD}_{\mathbb{R}} + \text{DC}$ every subset of a Polish space is in $\mathcal{S}(\infty) = \mathcal{S}(\Theta)$ by Proposition 9.23, and hence $\leq_{\mathcal{S}(\Theta)}$ would coincide with the reducibility \leq (without any definability condition on the reductions).

- THEOREM 15.16** ($\text{AD}_{\mathbb{R}} + \text{DC}$). (a) *The embeddability relation $\sqsubset_{\text{CT}}^{\Theta}$ is $\leq_{\mathbf{B}}^{\Theta}$ -complete for arbitrary quasi-orders on ${}^{\omega}2$.*
 (b) *Assume that Θ is regular. Then the relation $\sqsubset_{\text{CT}}^{\Theta}$ is also $\leq_{\mathbf{B}}^{\Theta}$ -invariantly universal for arbitrary quasi-orders on ${}^{\omega}2$.*

PROOF. By Proposition 9.23, under $\text{AD}_{\mathbb{R}} + \text{DC}$ every subset of ${}^{\omega}2$ is ∞ -Souslin and $\Xi = \Theta$. So it is enough to apply Theorems 14.3 and 14.8. \square

REMARK 15.17. As for Theorem 15.14, also in this case one can formulate and prove level-by-level versions of Theorem 15.16, namely: under $\text{AD}_{\mathbb{R}} + \text{DC}$ (which implies $\Xi = \Theta$ by Proposition 9.23), one can assign to *every* quasi-order R on ${}^{\omega}2$ a regular Souslin cardinal $\kappa_R < \Theta$ such that:

- (a) $R \leq_{\mathcal{S}(\kappa_R)} \sqsubset_{\text{CT}}^{\kappa_R}$;
 (b) $R \sim_{\mathbf{B}}^{\kappa_R} \sqsubset_{\sigma}^{\kappa_R}$ for some $\mathcal{L}_{\kappa_R^+ \kappa_R}$ -sentence σ all of whose models are combinatorial trees.

(Notice that in this case we necessarily have $\kappa_R < \Theta$ by Lemma 9.12.)

15.4. $L(\mathbb{R})$ -reducibility. In this section we present some results concerning the $L(\mathbb{R})$ -reducibility considered in [Hjo95, Hjo99, Hjo00] — see Example 1.10 and Section 14.3. We recall once more that the axiom $\text{AD}^{L(\mathbb{R})}$ used in Theorem 15.18 follows from both large cardinals and forcing axioms. Notice also that the pointclass $\mathbf{\Gamma}_1^2 := (\Sigma_1^2)^{L(\mathbb{R})}$ is closed (in V) under preimages and images of continuous functions because all such functions belong to $L(\mathbb{R})$.

THEOREM 15.18 (AC). *Assume $\text{AD}^{L(\mathbb{R})}$.*

- (a) *The relation $\sqsubset_{\text{CT}}^{\delta_1^2}$ is $\leq_{L(\mathbb{R})}$ -complete (and also $\leq'_{L(\mathbb{R})}$ -complete) for quasi-orders on ${}^{\omega}2$ belonging to $\mathbf{\Gamma}_1^2 := (\Sigma_1^2)^{L(\mathbb{R})}$.*
 (b) *For every $\mathbf{\Gamma}_1^2$ quasi-order R there is an $\mathcal{L}_{(\delta_1^2)^+ \delta_1^2}$ -sentence σ (in $L(\mathbb{R})$) such that $R \sim_{L(\mathbb{R})} \sqsubset_{\sigma}^{\delta_1^2}$.*

PROOF. Use Theorems 14.24, 14.27 and 14.25. \square

REMARKS 15.19. (i) When dealing with $L(\mathbb{R})$ -reducibility, the relations $\sqsubset_{\text{CT}}^{\kappa}$ and $\sqsubset_{\sigma}^{\kappa}$ (for $\kappa \in (\text{Card})^{L(\mathbb{R})}$ and $\sigma \in (\mathcal{L}_{\kappa^+ \kappa})^{L(\mathbb{R})}$) must be construed as $(\sqsubset_{\text{CT}}^{\kappa})^{L(\mathbb{R})}$ and $(\sqsubset_{\sigma}^{\kappa})^{L(\mathbb{R})}$, respectively — see Remark 14.26.

- (ii) As already noticed in the introduction after Theorem 1.16, Theorem 15.18 implies that every equivalence relation in $\mathbf{\Gamma}_1^2$, a quite large boldface pointclass in V which includes e.g. all projective levels, is $L(\mathbb{R})$ -reducible to a bi-embeddability relation $\approx_{\mathcal{L}}^{\kappa}$ (on structures of an appropriate uncountable size κ). This should be strongly contrasted with the case of the isomorphism relation: by Example 1.10 (see [Hjo00, Theorem 9.18]), there are even Σ_1^1 equivalence relations E such that $E \not\leq_{L(\mathbb{R})} \approx_{\mathcal{L}}^{\kappa}$ for *every* $\kappa \in (\text{Card})^{L(\mathbb{R})}$ (and hence also for every cardinal in V).
- (iii) In the statement of Theorem 15.18 we could replace $L(\mathbb{R})$ with *any* inner model W containing all the reals of the universe.

15.5. Reducibility in inner models for $\text{AD}_{\mathbb{R}}$. Assuming the existence of large cardinals, $L(\mathbb{R})$ models AD , but it does not satisfy $\text{AD}_{\mathbb{R}}$ because it has a Π_1^2 relation that cannot be uniformized (see page 14). Assuming even larger cardinals, it is possible to construct inner models satisfying stronger forms of determinacy, including $\text{AD}_{\mathbb{R}}$. These are models of AD^+ of the form $L(\Gamma)$ for a suitable selfdual pointclass Γ , and they extend $L(\mathbb{R})$ much like inner models for large cardinals extend L . We now give a very brief description of how these models look like, referring the interested reader to [Sar13] for a detailed exposition.

In what follows we work in some fixed model M of AD^+ (Definition 9.24). Let \leq_W be the Wadge reducibility relation on ${}^\omega\omega$, that is the prewellordering of⁴⁴ $\mathcal{P}({}^\omega\omega)$ defined by

$$A \leq_W B \Leftrightarrow A = f^{-1}(B) \text{ for some continuous } f: {}^\omega\omega \rightarrow {}^\omega\omega.$$

The rank of $A \subseteq {}^\omega\omega$ according to \leq_W is denoted by $\|A\|_W$, and the supremum of $\|A\|_W$ is the cardinal Θ of Definition 4.3. For $\alpha \leq \Theta$, the family

$$\mathcal{W}_\alpha := \{X \subseteq {}^\omega\omega \mid \|X\|_W < \alpha\}$$

is a selfdual pointclass, and $\mathcal{W}_\Theta = \mathcal{P}({}^\omega\omega)$. We now relativize these notions to an arbitrary subset of the Baire space: given $A \subseteq {}^\omega\omega$ let

$$\Theta(A) := \sup\{\|B\|_W \mid B \text{ is ordinal definable from reals and } A\},$$

and define the **Solovay sequence** $\langle \Theta_\alpha \mid \alpha < \Omega \rangle$ as follows:

$$\begin{aligned} \Theta_0 &:= \Theta(\emptyset) \\ \Theta_{\alpha+1} &:= \Theta(A) \text{ for some/any } A \text{ such that } \|A\|_W = \Theta_\alpha \\ \Theta_\lambda &:= \sup_{\alpha < \lambda} \Theta_\alpha. \end{aligned}$$

Define

$$\mathbb{M}_\alpha := L(\mathcal{W}_{\Theta_\alpha}), \quad (\alpha < \Omega).$$

It can be shown that:

- $\mathbb{M}_\alpha \models \text{AD}^+$
- $\mathcal{W}_{\Theta_\alpha} = (\mathcal{P}({}^\omega\omega))^{\mathbb{M}_\alpha}$, and hence $\mathbb{M}_\alpha \models "V = L(\mathcal{P}(\mathbb{R}))"$
- $\Theta_\alpha = (\Theta)^{\mathbb{M}_\alpha}$,
- $\mathbb{M}_{\alpha+1} = L(\mathbb{R} \cup \{A\})$ for some/any $A \subseteq {}^\omega\omega$ such that $\|A\|_W = \Theta_{\alpha+1}$,
- $\mathbb{M}_\alpha \models \text{AD}_{\mathbb{R}}$ if and only if α is a limit ordinal.

Since in $L(\mathbb{R})$ every set is ordinal definable from a real, then $\mathbb{M}_0 = L(\mathbb{R})$ and $(\Theta)^{\mathbb{M}_0} = \Theta_0$. If A_0 is any set of reals with Wadge rank Θ_0 , e.g. $A_0 = \mathbb{R}^\sharp$, then we can construct $\mathbb{M}_1 = L(\mathbb{R} \cup \{A_0\})$. More generally $\mathbb{M}_{n+1} = L(\mathbb{R} \cup \{A_n\})$ for some $A_n \subseteq {}^\omega\omega$ such that $\|A_n\|_W = \Theta_n$, and $\Theta^{\mathbb{M}_{n+1}} = \Theta_{n+1}$. The model \mathbb{M}_ω is $L(\mathcal{W}_{\Theta_\omega})$, where $\Theta_\omega := \sup_n \Theta_n$. Therefore the length of the Solovay sequence in \mathbb{M}_α is $\alpha + 1$, i.e. $\Omega^{\mathbb{M}_\alpha} = \alpha + 1$.

The discussion in the paragraph above takes place in some fixed model M of AD^+ . If M and N are arbitrary models of AD^+ with ${}^\omega\omega \subseteq M \cap N$ and $\Omega^M, \Omega^N \geq \omega$, then $\Theta_i^M = \Theta_i^N$ for all $i \leq \omega$, and hence $(\mathbb{M}_\omega)^M = L(\mathcal{W}_{\Theta_\omega})^M = L(\mathcal{W}_{\Theta_\omega})^N = (\mathbb{M}_\omega)^N$, and this model, which will simply be denoted by \mathbb{M}_ω is called the **minimal model** of $\text{AD}_{\mathbb{R}}$. The adjective minimal refers to the fact that any inner model of $\text{AD}_{\mathbb{R}}$ containing \mathbb{R} , must contain \mathbb{M}_ω .

The preceding paragraphs were under the assumption AD^+ . Stepping back in the AC-world, and assuming enough large cardinals (the existence of a Woodin limit of Woodin cardinals is overkill), it is possible to construct the minimal model \mathbb{M}_ω of $\text{AD}_{\mathbb{R}}$ even without any explicit determinacy assumption: one proves that the existence of such large cardinals implies that

⁴⁴In this section we could replace the Baire space ${}^\omega\omega$ with the Cantor space ${}^\omega 2$. In this other setup, the Wadge reducibility \leq_W coincides with the relation \leq_W^ω of Definition 3.8 in Section 3, page 23.

enough sets are determined, so that the sequence of the Θ_n is well-defined, and therefore \mathbb{M}_ω exists. Applying again Theorems 14.24, 14.27 and 14.25 in this setup, we now get:

THEOREM 15.20 (AC). *Assume that there is a Woodin cardinal which is limit of Woodin cardinals, let \mathbb{M}_ω be the minimal model of $\text{AD}_\mathbb{R}$, and let $\kappa := \Theta^{\mathbb{M}_\omega} = \Theta_\omega$.*

- (a) *The relation $\sqsubset_{\text{CT}}^\kappa$ is $\leq_{\mathbb{M}_\omega}$ -complete (and also $\leq'_{\mathbb{M}_\omega}$ -complete) for quasi-orders on ${}^\omega 2$ belonging to \mathbb{M}_ω .*
- (b) *For every quasi-order $R \in \mathbb{M}_\omega$ on ${}^\omega 2$ there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ (in \mathbb{M}_ω) such that $R \sim_{\mathbb{M}_\omega} \sqsubset_\sigma^\kappa$.*

Theorem 15.20 yields the following characterization of the quasi-orders on ${}^\omega 2$ belonging to \mathbb{M}_ω — recall that in Theorem 15.20 the embeddability $\sqsubset_{\text{CT}}^\kappa$ is construed as $(\sqsubset_{\text{CT}}^\kappa)^{\mathbb{M}_\omega}$, see Remark 14.26.

COROLLARY 15.21 (AC). *Assume that there is a Woodin cardinal which is limit of Woodin cardinals, let \mathbb{M}_ω be the minimal model of $\text{AD}_\mathbb{R}$, and let $\kappa := \Theta^{\mathbb{M}_\omega} = \Theta_\omega$. For every quasi-order R on ${}^\omega 2$ the following are equivalent:*

- (a) $R \in \mathbb{M}_\omega$;
- (b) $R \leq_{\mathbb{M}_\omega} \sqsubset_{\text{CT}}^\kappa$;
- (c) $R \sim_{\mathbb{M}_\omega} \sqsubset_\sigma^\kappa$ for some $\mathcal{L}_{\kappa+\kappa}$ -sentence σ in \mathbb{M}_ω .

In Theorem 15.20 and in Corollary 15.21 we may replace \mathbb{M}_ω with any other inner model of $\text{AD}_\mathbb{R}$ (containing all the reals of the universe).

16. Further completeness results

16.1. Representing arbitrary partial orders as embeddability relations. The methods developed in this paper yield also some purely combinatorial results, showing that embeddability relations can be quite complex. Recall from Example 1.3 that in ZFC, every partial order of size $\kappa = \aleph_1$ can be embedded into (the quotient order of) $\sqsubset_{\text{CT}}^\omega$: this is shown by proving that the relation $(\mathcal{P}(\omega), \subseteq^*)$ of inclusion modulo finite subsets on $\mathcal{P}(\omega)$ is Borel reducible to $\sqsubset_{\text{CT}}^\omega$, and then using Parovichenko's theorem to embed any partial order P of size \aleph_1 into the inclusion relation on $\mathcal{P}(\omega)/\text{Fin}$. If we assume enough choice, weak forms of the above fact can be obtained for all uncountable *small cardinals* — for larger cardinals see [MMR16].

PROPOSITION 16.1. *Let $\omega < \kappa \leq 2^{\aleph_0}$ and assume $\text{AC}_\kappa(\mathbb{R})$. Then every partial order P of size κ can be embedded into the quotient order of $\sqsubset_{\text{CT}}^\kappa$. In fact, for every such P there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ (all of whose models are combinatorial trees) such that the quotient order of \sqsubset_σ^κ is isomorphic to P .*

PROOF. Without loss of generality we may assume that P is $(\kappa, \trianglelefteq)$. We will now assign to each $\alpha < \kappa$ a combinatorial tree G_α of size κ such that for all $\alpha, \beta < \kappa$

$$(16.1) \quad \alpha \trianglelefteq \beta \Leftrightarrow G_\alpha \sqsubset G_\beta.$$

The map $\alpha \mapsto G_\alpha$ yields the desired embedding of P into the quotient order of $\sqsubset_{\text{CT}}^\kappa$.

Since by Theorem 1.1 the relation of equality on \mathbb{R} is (Borel) reducible to $\sqsubset_{\text{CT}}^\omega$ and $\kappa \leq 2^{\aleph_0}$, using $\text{AC}_\kappa(\mathbb{R})$ we can pick a sequence $\langle S_\delta \mid \delta < \kappa \rangle$ of countable combinatorial trees (with disjoint domains) such that $S_\delta \not\sqsubset S_{\delta'}$ for all distinct $\delta, \delta' < \kappa$, and fix an arbitrary element $x_\delta \in S_\delta$. Given $\alpha < \kappa$, set

$$C_\alpha := \{2 \cdot \gamma \mid \gamma < \kappa\} \cup \{2 \cdot \gamma + 1 \mid \gamma < \kappa, \gamma \trianglelefteq \alpha\}.$$

The combinatorial tree G_α is then (an isomorphic copy with domain κ of the graph) defined on the disjoint union

$$\{r_\alpha\} \uplus \bigcup \{S_\delta \mid \delta \in C_\alpha\}$$

by connecting the vertex r_α with each $x_\delta \in S_\delta$ (for $\delta \in C_\alpha$). Using the fact that embeddings cannot decrease degrees of vertices, that r_α has always degree κ and all other vertices of G_α have degree $\leq \omega < \kappa$, and that for all $\alpha, \beta < \kappa$

$$\alpha \trianglelefteq \beta \Leftrightarrow \{\gamma < \kappa \mid \gamma \trianglelefteq \alpha\} \subseteq \{\gamma < \kappa \mid \gamma \trianglelefteq \beta\} \Leftrightarrow C_\alpha \subseteq C_\beta,$$

we easily get that (16.1) is satisfied.

For the second part, one first check that the structure G_α admits a Scott sentence $\sigma_\alpha \in \mathcal{L}_{\kappa+\kappa}$ (see Remark 12.17(ii) for the definition). Such a Scott sentence is obtained by formalizing as in Section 12.1 the conjunction of the following statements (here we use that $\kappa \geq \omega_1$ and that each S_δ , being countable, can be described by its quantifier-free type, which is an $\mathcal{L}_{\omega_1\omega}^0$ -formula):

- (1) the structure is a combinatorial tree;
- (2) there is a unique vertex of degree $> \omega$;
- (3) all other vertices belong to a unique countable substructure of G_α which is isomorphic to some S_δ , and all these substructures are “disjoint” (vertices of different substructures are never connected by an edge);
- (4) the vertex with degree $> \omega$ is connected to a unique vertex in each of the substructures described in (3), and such vertex “corresponds” to $r_\delta \in S_\delta$;
- (5) for each $\delta < \kappa$, there is at most one substructure as in (3) which is isomorphic to S_δ ;
- (6) for each $\delta < \kappa$, there is a substructure as in (3) which is isomorphic to S_δ just in case $\delta \in C_\alpha$.

Then letting σ be the $\mathcal{L}_{\kappa+\kappa}$ sentence $\bigvee_{\alpha < \kappa} \sigma_\alpha$, we get that Mod_σ^κ is the closure under isomorphism (inside CT_κ) of the family $\{G_\alpha \mid \alpha < \kappa\}$. Therefore P and the quotient order of $\sqsubseteq_\sigma^\kappa$ are isomorphic by (16.1). \square

The first part of Proposition 16.1 may be improved to the following result, in which we denote by \subseteq_κ^* the relation $(\mathcal{P}(\kappa), \subseteq^*)$ of inclusion modulo bounded subsets on $\mathcal{P}(\kappa)$.

PROPOSITION 16.2. *Let $\omega < \kappa \leq 2^{\aleph_0}$ and assume $\text{AC}_\kappa(\mathbb{R})$. Then $\subseteq_\kappa^* \leq_{\mathbf{B}}^\kappa \sqsubseteq_{\text{CT}}^\kappa$.*

PROOF. Let T_0, T_1 , and $\langle S_\delta \mid \delta < \kappa \rangle$ be countable combinatorial trees such that:

- $S_\delta \not\sqsubseteq S_{\delta'}$ for all distinct $\delta, \delta' < \kappa$;
- $S_\delta \not\sqsubseteq T_i$ and $T_i \not\sqsubseteq S_\delta$ for every $\delta < \kappa$ and $i = 0, 1$;
- $T_0 \subseteq T_1$ but $T_1 \not\subseteq T_0$.

Combinatorial trees as above can easily be obtained by applying Theorem 1.1 to the Borel quasi-order on $\mathbb{R} \times \{0, 1\}$ defined by setting $(r, i) R (s, j) \Leftrightarrow r = s \wedge i \leq j$, and then using $\text{AC}_\kappa(\mathbb{R})$. For each $\delta < \kappa$ and $i = 0, 1$ fix arbitrary elements $x_\delta \in S_\delta$ and $y_i \in T_i$.

To each $X \subseteq \kappa$ we now associate a combinatorial tree G_X of size κ as follows. Fix distinct vertices r, r_α , and $p_{\alpha,\delta}$ for $\alpha, \delta < \kappa$. For each $\alpha, \delta < \kappa$ set

$$i_{\alpha,\delta}^X := \begin{cases} 1 & \text{if } \delta < \alpha \text{ or } \delta \in X \\ 0 & \text{otherwise.} \end{cases}$$

Append distinct copies of S_δ and $T_{i_{\alpha,\delta}^X}$ to the vertex $p_{\alpha,\delta}$ by connecting to it with an edge the (copies of) the distinguished vertices x_δ and $y_{i_{\alpha,\delta}^X}$ of S_δ and $T_{i_{\alpha,\delta}^X}$, respectively. Then append all the combinatorial trees obtained in this way (for a fixed α and arbitrary $\delta < \kappa$) to the vertex r_α by connecting to it with an edge the vertices $p_{\alpha,\delta}$. Finally, add an edge between r and each r_α : the resulting combinatorial tree is G_X .

The map associating (a suitable copy on κ of) the combinatorial tree G_X to each $X \in \mathcal{P}(\kappa)$ is $\kappa + 1$ -Borel (in fact: continuous): we claim that it also reduces \subseteq_κ^* to \sqsubseteq . Fix $X, Y \subseteq \kappa$ and assume first that there is $\beta < \kappa$ such that $\delta \in X \Rightarrow \delta \in Y$ for all $\beta \leq \delta < \kappa$. Consider the partial map e between G_X and G_Y sending r to itself, each r_α to $r_{\beta+\alpha}$, and, accordingly, each

$p_{\alpha,\delta}$ to $p_{\beta+\alpha,\delta}$. Since the choice of β ensures that $i_{\alpha,\delta}^X \leq i_{\beta+\alpha,\delta}^Y$ for all $\delta < \kappa$, the map e can be completed to an embedding of G_X into G_Y . Conversely, let e be an embedding of G_X into G_Y . Since r is the unique vertex in both G_X and G_Y having κ -many neighbors of degree κ , the embedding e must send r to itself and, consequently, r_0 to r_β for some $\beta < \kappa$. Our choice of the S_δ and T_i then implies that $e(p_{0,\delta}) = p_{\beta,\delta}$ and that e embeds $T_{i_{0,\delta}^X}$ into $T_{i_{\beta,\delta}^Y}$ (for all $\delta < \kappa$). Since $T_i \subseteq T_j \Leftrightarrow i \leq j$, this shows that $i_{0,\delta}^X \leq i_{\beta,\delta}^Y$ for all $\delta < \kappa$, i.e. that $\delta \in X \Rightarrow \delta \in Y$ for all $\beta \leq \delta < \kappa$: thus $X \subseteq^* Y$, as desired. \square

From Proposition 16.2 we in particular obtain that, under its assumptions, any partial order that can be embedded into (the quotient order of) $(\mathcal{P}(\kappa), \subseteq^*)$ can also be embedded into (the quotient order of) $\subseteq_{\text{CT}}^\kappa$. This applies to all partial orders of size κ , but also to many other interesting cases. For example, P. Schlicht and K. Thompson have recently verified (personal communication) that a straightforward adaptation to the uncountable case of Parovicenko's proof shows that under AC every linear order of size \aleph_{n+1} can be embedded into the quotient order of $(\mathcal{P}(\aleph_n), \subseteq^*)$.

In Theorem 16.4 below we provide a counterpart to Proposition 16.1 in the AD-world. The proof relies on the methods and results developed in this paper, and currently we do not see any simpler argument.

LEMMA 16.3 (AD + DC). *Let κ be a Souslin cardinal, and P be an arbitrary partial order of size κ . Then there is a κ -Souslin quasi-order R on ${}^\omega 2$ such that P embeds into the quotient order of R . If $\kappa < \delta_{\mathcal{S}(\kappa)}$ then R can be chosen so that its quotient order is isomorphic to P .*

PROOF. We can assume that P is of the form (κ, \leq) itself. Since ${}^\omega 2$ and ${}^\omega \omega$ are Borel isomorphic and $\mathcal{S}(\kappa)$ is closed under Borel preimages by Lemma 9.7, it is enough to find a quasi-order R as in the statement, but defined on ${}^\omega \omega$ instead of ${}^\omega 2$. Let ρ be an $\mathcal{S}(\kappa)$ -norm of length κ defined on a set $A \subseteq {}^\omega \omega$ belonging to $\mathcal{S}(\kappa)$ (which exists by Proposition 9.25(b)), and let $<$ be its strict part, i.e. for $x, y \in {}^\omega \omega$ set

$$x < y \Leftrightarrow x, y \in A \wedge \rho(x) < \rho(y).$$

Then $< \in \mathcal{S}(\kappa)$, and its rank function (which is just the $\mathcal{S}(\kappa)$ -norm ρ) is onto κ . Now consider the function $f: \kappa \times \kappa \rightarrow \mathcal{P}({}^\omega 2)$ defined by

$$f(\alpha, \beta) := \begin{cases} {}^\omega 2 & \text{if } \alpha \leq \beta, \\ \emptyset & \text{otherwise.} \end{cases}$$

By Moschovakis' first coding lemma [Mos09, Lemma 7D.5], there is an $\mathcal{S}(\kappa)$ set $C \subseteq {}^\omega \omega \times {}^\omega \omega \times {}^\omega 2$ which is a *choice set* for f , i.e. such that for all $x_1, x_2 \in {}^\omega \omega$, $y \in {}^\omega 2$, and $\alpha, \beta < \kappa$ the following hold:

- (1) $(x_1, x_2, y) \in C \Rightarrow x_1, x_2 \in A \wedge y \in f(\rho(x_1), \rho(x_2))$;
- (2) $f(\alpha, \beta) \neq \emptyset \Rightarrow \exists x_1, x_2 \in A \exists y \in {}^\omega 2 [\rho(x_1) = \alpha \wedge \rho(x_2) = \beta \wedge (x_1, x_2, y) \in C]$.

Let R be the quasi-order on ${}^\omega \omega$ defined by

$$z_1 R z_2 \Leftrightarrow z_1 = z_2 \vee [z_1, z_2 \in A \wedge \exists x_1, x_2 \in A \exists y \in {}^\omega 2 (\rho(x_1) = \rho(z_1) \wedge \rho(x_2) = \rho(z_2) \wedge (x_1, x_2, y) \in C)].$$

Then R is κ -Souslin by the closure properties of $\mathcal{S}(\kappa)$ (see Lemma 9.7), and it is straightforward to check that, by definition of C and the fact that it is a choice set for f , the map $\alpha \mapsto \{z \in A \mid \rho(z) = \alpha\}$ is an isomorphism between P and the quotient order of $R \upharpoonright (A \times A)$, and hence it is an embedding of P into the quotient order of the whole R .

The additional part concerning those κ which are smaller than $\delta_{\mathcal{S}(\kappa)}$ follows from the fact that in this case in the argument above we can let ρ be the rank function of any $\Delta_{\mathcal{S}(\kappa)}$ prewellordering of the whole ${}^\omega\omega$ of length κ , so that $A = {}^\omega\omega$. \square

THEOREM 16.4 (AD + DC). *Let κ be a Souslin cardinal. Then every partial order P of size κ can be embedded into the quotient order of \sqsubset_{CT}^κ . In fact, if $\kappa < \delta_{\mathcal{S}(\kappa)}$, then for every such P there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ (all of whose models are combinatorial trees) such that the quotient order of \sqsubset_σ^κ is isomorphic to P .*

Notice that by Proposition 9.25(c) the second part of Theorem 16.4 can be applied exactly when κ is such that $\mathcal{S}(\kappa)$ is not closed under coprojections, and thus, in particular, when κ is one of the projective cardinals δ_n^1 or, more generally, when κ is not a regular limit of Souslin cardinals.

PROOF. Let R be as in Lemma 16.3, and let $T \in \mathbb{T}_\kappa$ be such that $R = p[T]$. By Theorem 11.8, the function f_T defined in (11.5) reduces R to \sqsubset_{CT}^κ , and thus its quotient map

$$\hat{f}_T: {}^\omega\omega/R \rightarrow CT_\kappa/\approx_{CT}^\kappa, \quad [x]_{E_R} \mapsto [f_T(x)]_{\approx_{CT}^\kappa}$$

where \approx_{CT}^κ is the bi-embeddability relation on CT_κ , embeds the quotient order ${}^\omega\omega/R$ into the quotient order $CT_\kappa/\approx_{CT}^\kappa$. Therefore, composing the embedding of P into ${}^\omega\omega/R$ with \hat{f}_T gives the desired embedding.

If $\kappa < \delta_{\mathcal{S}(\kappa)}$, then by Lemma 16.3 there is a κ -Souslin quasi-order R on ${}^\omega 2$ whose quotient order is isomorphic to P . By Corollary 12.16, there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ (all of whose models are combinatorial trees) such that the quotient order of \sqsubset_σ^κ is isomorphic to ${}^\omega\omega/R$, and hence also to P . \square

Currently we do not know if it is possible to obtain an analogue of Proposition 16.2 in the AD-context.

16.2. Other model theoretic examples. In this section we consider some model theoretic variants of our results on the embeddability relation between combinatorial trees. We consider two kinds of variations: in the first one, we change the morphism between combinatorial trees under consideration by replacing embeddings with full homomorphisms, while in the second one we consider the embeddability relation again but we change the nature of the underlying structures. Of course these are only a few examples of the many interesting model theoretic variants of the problem considered in this paper (namely, the descriptive set-theoretic complexity of natural relations between uncountable structures) which would deserve some attention in the near future.

16.2.1. Changing the morphism: full homomorphisms between combinatorial trees. Recall that $\mathcal{L} = \{\mathbf{E}\}$ is the language of graphs consisting of a single binary relational symbol.

DEFINITION 16.5. Let $X = \langle X; \mathbf{E}^X \rangle$ and $Y = \langle Y; \mathbf{E}^Y \rangle$ be two \mathcal{L} -structures. A map $f: X \rightarrow Y$ is called **full homomorphism** between X and Y if for all $a, b \in X$

$$a \mathbf{E}^X b \Leftrightarrow f(a) \mathbf{E}^Y f(b).$$

Thus embeddings are just *injective* full homomorphisms. Moreover, since the identity function on an \mathcal{L} -structure is a full homomorphism, and since the composition of two full homomorphisms is still a full homomorphism, the binary relation defined below is reflexive and transitive, i.e. a quasi-order.

NOTATION. Given two \mathcal{L} -structures $X = \langle X; \mathbf{E}^X \rangle$ and $Y = \langle Y; \mathbf{E}^Y \rangle$, we set $X \sqsubseteq^h Y$ if and only if there exists a full homomorphism between X and Y .

Replacing \sqsubset with \sqsubset^h in Definitions 12.1 and 12.2 we get a corresponding notion of invariant universality for \sqsubset^h .

DEFINITION 16.6. Let \mathcal{C} be a class of quasi-orders, \mathcal{L} be the graph language, and κ be an infinite cardinal. Given an $\mathcal{L}_{\kappa+\kappa}$ -sentence τ , the relation $\sqsubset^h \upharpoonright \text{Mod}_\tau^\kappa$ is **invariantly universal for** \mathcal{C} if for every $R \in \mathcal{C}$ there is an $\mathcal{L}_{\kappa+\kappa}$ -sentence σ such that $\text{Mod}_\sigma^\kappa \subseteq \text{Mod}_\tau^\kappa$ and $R \sim \sqsubset^h \upharpoonright \text{Mod}_\sigma^\kappa$.

As usual, when the reducibility \leq is replaced by one of its restricted form \leq_* we speak of **\leq_* -invariant universality**.

We are now going to show that for every infinite cardinal κ and every $T \in \mathbb{T}_\kappa$, the relations \sqsubset and \sqsubset^h coincide on $\text{Mod}_{\sigma_T}^\kappa$, where σ_T is as in Corollary 12.12. To prove this, we will use the following variant of [FMR11, Proposition 3.10], which can be proved in the same way.

LEMMA 16.7. *Suppose that G, G' are combinatorial trees and $f: G \rightarrow G'$ is a full homomorphism. If $a, b \in G$ are distinct and $f(a) = f(b)$, then a and b have both degree 1 in G and have geodesic distance 2 from each other (i.e. they share their unique adjacent vertex).*

This essentially shows that a full homomorphism between combinatorial trees is already almost injective, and allows us to prove the following.

THEOREM 16.8. *Let κ be an uncountable cardinal, $T \in \mathbb{T}_\kappa$ be such that $R = p[T]$ is a quasi-order, and σ_T be as in Corollary 12.12. Then the relation $\sqsubset^h \upharpoonright \text{Mod}_{\sigma_T}^\kappa$ coincides with $\sqsubset_{\sigma_T}^\kappa$ (that is, with the embeddability relation $\sqsubset \upharpoonright \text{Mod}_{\sigma_T}^\kappa$).*

PROOF. First recall that by Corollary 12.12 the set $\text{Mod}_{\sigma_T}^\kappa$ is the closure under isomorphism of $\text{ran}(f_T)$, where f_T is as in (11.5). Since embeddings are in particular full homomorphisms, it is enough to show that for every $x, y \in {}^\omega 2$

$$f_T(x) \sqsubset^h f_T(y) \Rightarrow f_T(x) \sqsubset f_T(y).$$

Let j be a full homomorphism between $f_T(x) = \mathbb{G}_{\Sigma_T(x)}$ and $f_T(y) = \mathbb{G}_{\Sigma_T(y)}$. By Lemma 16.7, its restriction to $G_0 \cup F(\mathbb{G}_{\Sigma_T(x)})$ (see Definition 10.2 and the notation introduced in (10.7a)–(10.7g) of Section 10.2) is injective, and hence an embedding. Arguing as in the second half of the proof of Theorem 11.8(a), we thus get that $j \upharpoonright (G_0 \cup F(\mathbb{G}_{\Sigma_T(x)}))$ is an embedding of the subgraph of $f_T(x)$ with domain $G_0 \cup F(\mathbb{G}_{\Sigma_T(x)})$ into the subgraph of $f_T(y)$ with domain $G_0 \cup F(\mathbb{G}_{\Sigma_T(y)})$. By the proof of Theorem 11.8(a) (see Remark 11.9), this implies that $x R y$, whence $f_T(x) \sqsubset f_T(y)$ by Theorem 11.8(a) again. \square

REMARK 16.9. By adapting in the obvious way Definitions 16.5 and 16.6 to the language $\bar{\mathcal{L}}$ of *ordered* combinatorial trees (see Section 13), one can easily check that in Theorem 16.8 we can also replace σ_T with the sentence $\bar{\sigma}_T$ from Corollary 13.9. In fact, the ordered combinatorial trees $\bar{f}_T(x)$ and $\bar{f}_T(y)$ (where \bar{f}_T is as in (13.2)) already have domain $G_0 \cup F(\mathbb{G}_{\Sigma_T(x)})$ and $G_0 \cup F(\mathbb{G}_{\Sigma_T(y)})$, respectively. Thus any full homomorphism between $\bar{f}_T(x)$ and $\bar{f}_T(y)$ is an embedding by Lemma 16.7 (which holds for ordered combinatorial trees as well, since these are $\bar{\mathcal{L}}$ -expansions of combinatorial trees).

Theorem 16.8 and Remark 16.9 show that in all the completeness and invariant universality results from Sections 11–15 and 16.1 we could systematically replace the embeddability relation \sqsubset with the one induced by full homomorphisms \sqsubset^h .

16.2.2. *Changing the structures: partial orders and lattices.* At the end of [LR05, Section 3.1], Louveau and Rosendal provided a continuous map from countable connected graphs to lattices (viewed as partial orders) which reduces the embeddability relation to itself. Such a construction can straightforwardly be adapted to the uncountable case. For technical reasons which will be clear shortly we have to modify the construction a bit.

DEFINITION 16.10. Let κ be any infinite cardinal. To any graph G on κ we associate the following lattice $L_G = (\kappa, \preceq_G)$:

- $0 \preceq_G \alpha \preceq_G 1$ for every $\alpha \in \kappa$
- if in G there is an edge between $\alpha, \beta \in \kappa$, then both $2 + (2 \cdot \alpha) \preceq_G 2 + (2 \cdot \langle \alpha, \beta \rangle + 1)$ and $2 + (2 \cdot \beta) \preceq_G 2 + (2 \cdot \langle \alpha, \beta \rangle + 1)$ (where $\langle \cdot, \cdot \rangle$ is the pairing function from (2.1));
- no other \preceq_G -relation holds.

Notice that the L_G 's are actually *complete* lattices.

REMARK 16.11. The map $G \mapsto L_G$ is continuous when both spaces of models (namely, the space of graphs of size κ and the space of lattices of size κ) are endowed with the same topology τ_p or τ_b . Our modification of the original Louveau-Rosendal construction is required to obtain continuity with respect to the topology τ_p .

THEOREM 16.12. *The map $G \mapsto L_G$ from Definition 16.10 reduces the embeddability (respectively, the isomorphism) relation between connected graphs to the embeddability (respectively, the isomorphism) relation between lattices.*

PROOF. This is just a minor variation of the proof of [LR05, Theorem 3.3]. If j is an embedding between the connected graphs G and G' , then the map defined by

$$\begin{aligned} 0 &\mapsto 0 \\ 1 &\mapsto 1 \\ 2 + (2 \cdot \alpha) &\mapsto 2 + (2 \cdot j(\alpha)) \\ 2 + (2 \cdot \langle \alpha, \beta \rangle + 1) &\mapsto 2 + (2 \cdot \langle j(\alpha), j(\beta) \rangle + 1) \end{aligned}$$

is an embedding of L_G into $L_{G'}$.

Conversely, let j be an embedding of L_G into $L_{G'}$. Notice that in both L_G and $L_{G'}$ we have that an ordinal $1 \neq \alpha \in \kappa$ is an immediate predecessor of the maximum 1 if and only if it is odd (here we use the fact that G and G' are connected). Since j is an embedding, we thus get that for every $\alpha \in \kappa$ there is $\gamma_\alpha \in \kappa$ such that $j(2 + (2 \cdot \alpha)) = 2 + (2 \cdot \gamma_\alpha)$. Since $\alpha, \beta \in \kappa$ are connected by an edge in G if and only if $2 + (2 \cdot \alpha)$ and $2 + (2 \cdot \beta)$ share a \preceq_G -successor distinct from the maximum 1 (and the same is true for G'), it follows that the map $\alpha \mapsto \gamma_\alpha$ is an embedding of G into G' .

The same proof works for the case of the isomorphism relation, so we are done. \square

Theorem 16.12 and Remark 16.11 show that in all the completeness results from Sections 11–15 and 16.1 we could systematically replace combinatorial trees with (complete) lattices, and hence, in particular, with partial orders. Moreover, both Theorem 16.12 and Remark 16.11 remain true if the lattice L_G associated to the graph G is construed as a bounded lattice in the algebraic sense, that is as a structure in the language consisting of two binary function symbols (the join and the meet operations \vee and \wedge) and two constant symbols (the minimum and the maximum 0 and 1). Thus we could also further replace combinatorial trees with such algebraic structures.

16.3. Isometry and isometric embeddability between complete metric spaces of density character κ . Recall from Section 7.2.3 the standard Borel κ -space \mathfrak{M}_κ of (codes for) complete metric spaces of density character κ , and its subspaces \mathfrak{D}_κ and \mathfrak{U}_κ consisting of, respectively, discrete and ultrametric spaces. In this section we will provide some informations on the complexity of the isometry relation \cong^i , and the isometric embeddability relation \sqsubseteq^i , which are both κ -analytic quasi-orders on \mathfrak{M}_κ , and on their restrictions to \mathfrak{D}_κ and \mathfrak{U}_κ . Some of the results concerning \cong^i already appeared in [MR], but we include them in our presentation as well for the reader's convenience.

16.3.1. *The discrete case.* We first consider the case of *discrete* metric spaces of density character (equivalently, of size) κ . Fix strictly positive $r_0, r_1 \in \mathbb{R}$ such that $r_0 < r_1 \leq 2r_0$. To each graph G on κ we associate the discrete metric space D_G on κ with distance d_G defined by

$$d_G(\alpha, \beta) := \begin{cases} 0 & \text{if } \alpha = \beta \\ r_0 & \text{if } \alpha \neq \beta \text{ and } \alpha \text{ and } \beta \text{ are adjacent in } G \\ r_1 & \text{if } \alpha \neq \beta \text{ and } \alpha \text{ and } \beta \text{ are not adjacent in } G. \end{cases}$$

Our choice of r_0 and r_1 guarantees that the triangular inequality is satisfied by d_G . Since the space D_G is already defined on κ , it can canonically be identified with its code $x_G \in \mathfrak{D}_\kappa$ obtained by setting, as in (7.6), $x_G(\alpha, \beta, q) = 1 \Leftrightarrow d_G(\alpha, \beta) < q$ for every $\alpha, \beta < \kappa$ and $q \in \mathbb{Q}^+$.

REMARK 16.13. Let $\text{Mod}_{\text{GRAPH}}^\kappa$ be the space of all graphs on κ . Then the map

$$(16.2) \quad \theta_D: \text{Mod}_{\text{GRAPH}}^\kappa \rightarrow \mathfrak{D}_\kappa \subseteq \mathfrak{M}_\kappa, \quad G \mapsto x_G$$

is continuous when both $\text{Mod}_{\text{GRAPH}}^\kappa$ and \mathfrak{M}_κ are endowed with the same topology τ_p or τ_b .

LEMMA 16.14. *The map θ_D from (16.2) simultaneously reduces \cong to \cong^i and \sqsubseteq to \sqsubseteq^i .*

PROOF. It is easy to check that given any $G, G' \in \text{Mod}_{\text{GRAPH}}^\kappa$ and any $\varphi: \kappa \rightarrow \kappa$, the map φ is an isomorphism (respectively, an embedding) between G and G' if and only if it is an isometry (respectively, an isometric embedding) between D_G and $D_{G'}$. \square

This yields the following lower bounds for the complexity of $\cong^i \upharpoonright \mathfrak{D}_\kappa$ and $\sqsubseteq^i \upharpoonright \mathfrak{D}_\kappa$. As usual, we endow both $\text{Mod}_{\text{GRAPH}}^\kappa$ and \mathfrak{M}_κ with the bounded topology and \mathfrak{D}_κ with the induced relative topology. Moreover, to simplify the notation we denote by $\cong_{\text{GRAPH}}^\kappa$ and $\sqsubseteq_{\text{GRAPH}}^\kappa$ the isomorphism and the embeddability relation on $\text{Mod}_{\text{GRAPH}}^\kappa$, respectively.

THEOREM 16.15. *Let κ be any infinite cardinal.*

- (a) $\cong_{\text{GRAPH}}^\kappa \leq_{\mathbf{B}}^\kappa \cong^i \upharpoonright \mathfrak{D}_\kappa$ and $\sqsubseteq_{\text{GRAPH}}^\kappa \leq_{\mathbf{B}}^\kappa \sqsubseteq^i \upharpoonright \mathfrak{D}_\kappa$.
- (b) *The relation $\sqsubseteq^i \upharpoonright \mathfrak{D}_\kappa$ is $\leq_{\mathbf{B}}^\kappa$ -complete for the class of κ -Souslin quasi-orders on Polish or standard Borel spaces.*
- (c) (AC) *If $\kappa \leq 2^{\aleph_0}$ and there is an $\mathcal{S}(\kappa)$ -code for κ (which is always the case for $\kappa = \omega$, $\kappa = \omega_1$, and $\kappa = 2^{\aleph_0}$), then $\sqsubseteq^i \upharpoonright \mathfrak{D}_\kappa$ is $\leq_{\mathcal{S}(\kappa)}$ -complete for κ -Souslin quasi-orders on Polish or standard Borel spaces.*
- (d) (AD + DC) *If κ is a Souslin cardinal, then $\sqsubseteq^i \upharpoonright \mathfrak{D}_\kappa$ is $\leq_{\mathcal{S}(\kappa)}$ -complete for κ -Souslin quasi-orders on Polish or standard Borel spaces.*
- (e) *Let $W \supseteq \mathbb{R}$ be an inner model and let $\kappa \in \text{Card}^W$. Then $\sqsubseteq^i \upharpoonright \mathfrak{D}_\kappa$ is \leq_W -complete for quasi-orders in $\mathcal{S}_W(\kappa)$.*

PROOF. (a) By Lemma 16.14, this is witnessed by the map from (16.2), which is τ_b -continuous (and hence $\kappa + 1$ -Borel) by Remark 16.13.

(b) We argue as in the proof of Theorem 14.3. Let $R = p[T]$ be an arbitrary κ -Souslin quasi-order on ${}^\omega 2$, and consider the map $\theta_D \circ f_T: {}^\omega 2 \rightarrow \mathfrak{D}_\kappa \subseteq \mathfrak{M}_\kappa$, where f_T is as in (11.5) and θ_D is as in (16.2). By Remark 16.13, such map is continuous when both ${}^\omega 2$ and \mathfrak{M}_κ are endowed with the product topology, and hence it is (effective) weakly $\kappa + 1$ -Borel. Moreover, it reduces R to \sqsubseteq^i by Theorem 11.8 and Lemma 16.14, so we are done.

(c)–(d) Argue as in the proof of Theorem 14.6, using again the fact that $\theta_D \circ f_T: {}^\omega 2 \rightarrow \mathfrak{D}_\kappa$ is continuous when both spaces are endowed with the product topology and that it reduces the κ -Souslin quasi-order $R = p[T]$ to \sqsubseteq^i .

(e) Use Theorem 14.24 together with the fact that the map θ_D from (16.2) is definable in W and that Lemma 16.14, which is a theorem of ZF, is true in W as well. \square

By Theorem 16.15, in all the completeness results from Sections 14, 15, and 16.1 we may systematically replace $\sqsubset_{\text{CT}}^\kappa$ with $\sqsubseteq^i \restriction \mathfrak{D}_\kappa$; a sample of (stronger forms) of these variants will be presented in Section 16.3.2 (see Theorems 16.22, 16.23 and Remark 16.24). Moreover, from Theorem 16.15(a) it follows that both the isometry relation \cong^i and the isometric embeddability relation \sqsubseteq^i are consistently as complicated as possible, even when restricted to \mathfrak{D}_κ .

COROLLARY 16.16. (a) *Assume $V = L$, and let $\kappa = \lambda^+$ be such that $\lambda^\omega = \lambda$ (equivalently: λ is either a successor cardinal or a limit cardinal of uncountable cofinality). Then the relation $\cong^i \restriction \mathfrak{D}_\kappa$ is $\leq_{\mathbf{B}}^\kappa$ -complete for the collection of all κ -analytic equivalence relations on standard Borel κ -spaces.*

(b) *Assume AC and let κ be a weakly compact⁴⁵ cardinal. Then the relation $\sqsubseteq^i \restriction \mathfrak{D}_\kappa$ is $\leq_{\mathbf{B}}^\kappa$ -complete for the collection of all κ -analytic quasi-orders on standard Borel κ -spaces.*

PROOF. (a) In [HK15, Corollary 2.15] it is proved that under our assumptions there is a first-order theory T such that the isomorphism relation on the collection Mod_T^κ of models of size κ of T is $\leq_{\mathbf{B}}^\kappa$ -complete for κ -analytic equivalence relations on ${}^\kappa 2$. This can be extended to κ -analytic equivalence relations on arbitrary standard Borel κ -spaces by adapting the proof of [MR13, Lemma 6.8] to our context, using the fact that any standard Borel κ -space is $\kappa + 1$ -Borel isomorphic to some $B \in \mathbf{B}_{\kappa+1}({}^\kappa 2, \tau_b)$. By [Hod93, Theorem 5.5.1], T can be interpreted in the theory of graphs: more precisely, there is a τ_b -continuous map $\text{Mod}_T^\kappa \rightarrow \text{Mod}_{\text{GRAPH}}^\kappa$ which simultaneously reduces $\cong \restriction \text{Mod}_T^\kappa$ to $\cong_{\text{GRAPH}}^\kappa$ and $\sqsubset \restriction \text{Mod}_T^\kappa$ to $\sqsubset_{\text{GRAPH}}^\kappa$. In particular, this shows that under our assumptions $\cong_{\text{GRAPH}}^\kappa$ is $\leq_{\mathbf{B}}^\kappa$ -complete for κ -analytic equivalence relations on ${}^\kappa 2$ as well. Thus the result follows from Theorem 16.15(a).

(b) The argument is similar to that of part (a). In [MR13, Corollary 9.5] it is proved that under our assumptions the embeddability relation on *generalized trees*⁴⁶ of size κ is $\leq_{\mathbf{B}}^\kappa$ -complete for κ -analytic quasi-orders on standard Borel κ -spaces. By the above argument, this implies that the same is true for $\sqsubset_{\text{GRAPH}}^\kappa$. Thus the result follows again from Theorem 16.15(a). \square

Notice that we can also complement Theorem 16.15(a) and determine the exact complexity with respect to $\leq_{\mathbf{B}}^\kappa$ of the relations $\cong^i \restriction \mathfrak{D}_\kappa$ and $\sqsubseteq^i \restriction \mathfrak{D}_\kappa$.

THEOREM 16.17. *Let κ be any infinite cardinal. Then $\cong^i \restriction \mathfrak{D}_\kappa \sim_{\mathbf{B}}^\kappa \cong_{\text{GRAPH}}^\kappa$ and $\sqsubseteq^i \restriction \mathfrak{D}_\kappa \sim_{\mathbf{B}}^\kappa \sqsubset_{\text{GRAPH}}^\kappa$.*

PROOF. By Theorem 16.15(a), we only need to show that $\cong^i \restriction \mathfrak{D}_\kappa \leq_{\mathbf{B}}^\kappa \cong_{\text{GRAPH}}^\kappa$ and $\sqsubseteq^i \restriction \mathfrak{D}_\kappa \leq_{\mathbf{B}}^\kappa \sqsubset_{\text{GRAPH}}^\kappa$. Consider the countable language $\hat{\mathcal{L}} := \{P_q \mid q \in \mathbb{Q}^+\}$, where each P_q is a binary relational symbol. To each $x \in \mathfrak{D}_\kappa$ associate the $\hat{\mathcal{L}}$ -structure \mathcal{A}_x on κ defined by setting for each $q \in \mathbb{Q}^+$ and $\alpha, \beta < \kappa$

$$P_q^{\mathcal{A}_x}(\alpha, \beta) \Leftrightarrow x(\alpha, \beta, q) = 1.$$

Then the map $x \mapsto \mathcal{A}_x$ is continuous when both \mathfrak{D}_κ and $\text{Mod}_{\hat{\mathcal{L}}}^\kappa$ are endowed with the bounded topology, and moreover for every $x, y \in \mathfrak{D}_\kappa$

$$x \cong^i y \Leftrightarrow \mathcal{A}_x \cong \mathcal{A}_y \quad \text{and} \quad x \sqsubseteq^i y \Leftrightarrow \mathcal{A}_x \sqsubset \mathcal{A}_y.$$

Indeed, using the notation from Section 7.2.3, a map $f: \kappa \rightarrow \kappa$ is an isometry (respectively, an isometric embedding) between $M_x = (\kappa, d_x)$ and $M_y = (\kappa, d_y)$ if and only if it is an isomorphism (respectively, an embedding) between \mathcal{A}_x and \mathcal{A}_y . This shows that $\cong^i \restriction \mathfrak{D}_\kappa \leq_{\mathbf{B}}^\kappa \cong \restriction \text{Mod}_{\hat{\mathcal{L}}}^\kappa$ and $\sqsubseteq^i \restriction \mathfrak{D}_\kappa \leq_{\mathbf{B}}^\kappa \sqsubset \restriction \text{Mod}_{\hat{\mathcal{L}}}^\kappa$. Since as discussed in the proof of Corollary 16.16(a) we have $\cong \restriction \text{Mod}_{\hat{\mathcal{L}}}^\kappa \leq_{\mathbf{B}}^\kappa \cong_{\text{GRAPH}}^\kappa$ and $\sqsubset \restriction \text{Mod}_{\hat{\mathcal{L}}}^\kappa \leq_{\mathbf{B}}^\kappa \sqsubset_{\text{GRAPH}}^\kappa$ by [Hod93, Theorem 5.5.1], we are done. \square

⁴⁵[MMR16] shows that the same is true for any cardinal κ satisfying the equality $\kappa^{<\kappa} = \kappa$.

⁴⁶We call **generalized tree** any partial order $T = (T, \leq_T)$ that is a tree in the model theoretic sense, that is: for every $t \in T$, the set $\{t' \in T \mid t' \leq_T t\}$ is linearly ordered by \leq_T .

16.3.2. *The ultrametric case.* In this section we consider the case of complete *ultrametric* spaces of density character κ and prove some strengthenings of the results from the previous section. Most of the definitions and results are direct generalizations to the uncountable context of the material from [CMMR] — see also [MR].

Let $U = (U, d_U)$ be an ultrametric space. Then by [MR, Lemma 2.20] for every non-negative $r \in \mathbb{R}$ and every dense $D \subseteq U$

$$(16.3) \quad \exists x, y \in U (d_U(x, y) = r) \Leftrightarrow \exists x, y \in D (d_U(x, y) = r).$$

It follows that if U has density character κ , then its set of (realized) distances

$$D(U) := \{r \in \mathbb{R} \mid \exists x, y \in U (d_U(x, y) = r)\}$$

has size at most κ . Conversely, given any set $A \subseteq \mathbb{R}$ of non-negative reals containing 0 and of size $\leq \kappa$, we can form a complete⁴⁷ ultrametric space $U(A) = (U_A, d_A)$ of density character κ such that $D(U(A)) = A$ by setting $U_A := \{(r, \alpha) \mid r \in A, \alpha < \kappa\}$ and for every $r, r' \in A$ and $\alpha, \beta < \kappa$

$$(16.4) \quad d_A((r, \alpha), (r', \beta)) := \begin{cases} 0 & \text{if } r = r' \text{ and } \alpha = \beta \\ \max\{r, r'\} & \text{otherwise.} \end{cases}$$

Thus we have that a set $A \subseteq \mathbb{R}$ is the set of (realized) distances of a complete ultrametric space of density character κ if and only if $0 \in A \subseteq [0; +\infty)$ and A has size $\leq \kappa$. This suggests to consider the following subclasses of \mathfrak{U}_κ (where for simplicity we denote by U_x the complete space coded by $x \in \mathfrak{U}_\kappa$ — see Section 7.2.3).

DEFINITION 16.18. Let κ be an infinite cardinal and $A \subseteq \mathbb{R}$ be such that $0 \in A \subseteq [0; +\infty)$ and $|A| \leq \kappa$. Then we set

$$\mathfrak{U}_\kappa(A) := \{x \in \mathfrak{U}_\kappa \mid D(U_x) \subseteq A\}$$

and

$$\mathfrak{U}_\kappa^*(A) := \{x \in \mathfrak{U}_\kappa \mid D(U_x) = A\}.$$

Using (16.3), it is easy to see that both $\mathfrak{U}_\kappa(A)$ and $\mathfrak{U}_\kappa^*(A)$ are effective $\kappa+1$ -Borel subsets of \mathfrak{U}_κ : thus since the latter is a standard Borel κ -space, $\mathfrak{U}_\kappa(A)$ and $\mathfrak{U}_\kappa^*(A)$ are standard Borel κ -spaces as well. Notice also that the spaces $\mathfrak{U}_\kappa^{(*)}(A)$ are always nonempty because $U(A) \in \mathfrak{U}_\kappa^*(A) \subseteq \mathfrak{U}_\kappa(A)$. However, the sets A that can be considered in these constructions may vary from a model of set theory to another: for example, in models of AD only countable A 's can satisfy the conditions in Definition 16.18 because there are no uncountable well-orderable subsets of \mathbb{R} by the (ω) -PSP.

We are now going to show that when a set A as in Definition 16.18 is ill-founded with respect to the usual ordering of \mathbb{R} , then the restriction of the isometry relation \cong^i and of the isometric embeddability relation \sqsubseteq^i to $\mathfrak{U}_\kappa^*(A)$ are quite complicated. This in particular yields strengthenings of the results from Section 16.3.1 because when 0 is not an accumulation point of A we have

$$\mathfrak{U}_\kappa^*(A) \subseteq \mathfrak{U}_\kappa(A) \subseteq \mathfrak{U}_\kappa \cap \mathfrak{D}_\kappa.$$

The following construction essentially corresponds to the special case $\alpha = \omega$ in [MR, Section 3], where it is shown that it yields a reduction from isomorphism to isometry; here we are going to show that the same construction yields also a reduction of embeddability to isometric embeddability. Fix a strictly decreasing sequence $\vec{r} = \langle r_n \mid n \in \omega \rangle$ of positive real numbers. Given a rooted combinatorial tree G on κ , define a partial order \preceq_G on κ by setting

$$\alpha \preceq_G \beta \Leftrightarrow \text{the unique path in } G \text{ connecting the root } r \text{ of } G \text{ to } \beta \text{ passes through } \alpha.$$

Define a metric $d_G^\vec{r}$ on κ by setting $d_G^\vec{r}(\alpha, \beta) := r_{n(\alpha, \beta)}$, where

$$n(\alpha, \beta) := \max\{l_G(\gamma) \mid \gamma \in \kappa, \gamma \preceq_G \alpha, \beta\}$$

⁴⁷Completeness follows automatically from the fact that $U(A)$ is discrete.

and $l_G(\gamma)$ is the length of the path connecting r to γ , and then consider the completion $U_G^{\vec{r}} = (U_G, d_G^{\vec{r}})$ of the space $(\kappa, d_G^{\vec{r}})$. It is clear from the construction the $U_G^{\vec{r}}$ is a complete ultrametric space of density character κ , and that its canonical code $u_G^{\vec{r}} \in \mathfrak{U}_\kappa$ defined by setting for every $\alpha, \beta < \kappa$ and $q \in \mathbb{Q}^+$

$$u_G^{\vec{r}}(\alpha, \beta, q) = 1 \Leftrightarrow d_G^{\vec{r}}(\alpha, \beta) < q$$

belongs to $\mathfrak{U}_\kappa(A)$ for any A as in Definition 16.18 containing all the r_n 's.

REMARK 16.19. Let RCT_κ be the space of (codes for) all rooted combinatorial trees on κ , A be as in Definition 16.18 and ill-founded, and \vec{r} be a strictly decreasing sequence of elements of A . Then the map

$$(16.5) \quad \theta_U^{\vec{r}}: \text{RCT}_\kappa \rightarrow \mathfrak{U}_\kappa(A) \subseteq \mathfrak{M}_\kappa, \quad G \mapsto u_G^{\vec{r}}$$

is continuous when both RCT_κ and $\mathfrak{U}_\kappa(A) \subseteq \mathfrak{M}_\kappa$ are endowed with the same topology τ_p or τ_b .

LEMMA 16.20. *The map $\theta_U^{\vec{r}}$ from (16.5) simultaneously reduces \cong to \cong^i and \sqsubset to \sqsubset^i .*

PROOF. If $\inf \vec{r} > 0$, then $U_G^{\vec{r}} = (\kappa, d_G^{\vec{r}})$ and one can simply use the straightforward adaptation to the uncountable context of the proof of [CMMR, Theorem 5.2] — the construction provided in this paper coincide with the one from [CMMR, Section 5.1] once we identify each element of RCT_κ with any of its isomorphic copies having domain a subset of ${}^{<\omega}\kappa$ closed under subsequences and root the empty sequence \emptyset .

If instead $\inf \vec{r} = 0$, the construction used in this paper is slightly different from the ones used in [GK03, Theorem 4.4] and [LR05, Proposition 4.2]. In fact, in this case $U_G^{\vec{r}}$ may be identified with a space on $G \uplus [G]$, where

$$[G] := \{b \in {}^\omega \kappa \mid b(0) = r \wedge \forall n \in \omega (b(n) \text{ } G \text{ } b(n+1))\}$$

is the set of all infinite ω -branches through G starting from its root r , while in [GK03] and [LR05] the authors (essentially) considered only $G \in \text{RCT}_\kappa$ without terminal vertices and associated to each of them only the subspace $[G]$ of $U_G^{\vec{r}}$. Although the two constructions are very close to each other (and in fact essentially equivalent), for the reader's convenience we give a sketch of the proof of the desired result based only on our new construction.

Let $G, G' \in \text{RCT}_\kappa$. We first deal with isomorphism and isometry, giving just the main ideas and referring the reader to [MR, Section 3] for more details. By construction, any isomorphism (respectively, embedding) between G and G' naturally extends to an isometry (respectively, isometric embedding) between $U_G^{\vec{r}} = G \uplus [G]$ and $U_{G'}^{\vec{r}} = G' \uplus [G']$. For the other direction, first notice that the (adaptation to the uncountable context of the) proof of [CMMR, Theorem 5.2] shows that if there exists an isometry (respectively, an isometric embedding) ψ between $(\kappa, d_G^{\vec{r}})$ and $(\kappa, d_{G'}^{\vec{r}})$, then G is isomorphic to (respectively, embeds into) G' . It is thus enough to show that if there is an isometry (respectively, an isometric embedding) $\varphi: U_G^{\vec{r}} \rightarrow U_{G'}^{\vec{r}}$, then there is also an isometry (respectively, an isometric embedding) $\psi: (\kappa, d_G^{\vec{r}}) \rightarrow (\kappa, d_{G'}^{\vec{r}})$. Let $\varphi: U_G^{\vec{r}} \rightarrow U_{G'}^{\vec{r}}$ be a metric-preserving map, and assume first that it is surjective, and therefore hence an isometry. Since by construction a point is isolated in $U_G^{\vec{r}}$ if and only if it belongs to G (and the same is true when replacing G with G'), then $\varphi(G) = G'$: thus $\psi := \varphi \upharpoonright G$ is an isometry between $(\kappa, d_G^{\vec{r}})$ and $(\kappa, d_{G'}^{\vec{r}})$ and we are done.

Assume now that φ is just an isometric embedding (i.e. not necessarily surjective). Then for some $\alpha \in G$ we may have $\varphi(\alpha) = b_\alpha \in [G']$. However, since α is isolated in $U_G^{\vec{r}}$, then b_α must be isolated in the range of φ : this implies that there is $n_\alpha \in \omega$ such that $\text{ran } \varphi$ does not contain any $\gamma \in G'$ with $b_\alpha(n_\alpha) \preceq_{G'} \gamma$ nor any $b' \in [G']$ such that $b_\alpha(n_\alpha) = b'(n_\alpha)$. By our choice of n_α , it easily follows that for every $\beta \in G$ distinct from α we have $d_{G'}^{\vec{r}}(b_\alpha, \varphi(\beta)) = d_{G'}^{\vec{r}}(b_\alpha(n_\alpha), \varphi(\beta))$.

Thus the map $\psi: G \rightarrow G'$ defined by setting for every $\alpha \in G$

$$\psi(\alpha) := \begin{cases} \varphi(\alpha) & \text{if } \varphi(\alpha) \in G' \\ b_\alpha(n_\alpha) & \text{if } \varphi(\alpha) \in [G'] \end{cases}$$

is a well-defined isometric embedding between $(\kappa, d_G^\vec{r})$ and $(\kappa, d_{G'}^\vec{r})$, so we are done. \square

Similarly to the discrete case, Remark 16.19 and Lemma 16.20 yields the following lower bounds for the complexity of $\cong^i \upharpoonright \mathfrak{U}_\kappa(A)$ and $\sqsubseteq^i \upharpoonright \mathfrak{U}_\kappa(A)$. As usual, we endow both RCT_κ and \mathfrak{M}_κ with the bounded topology and $\mathfrak{U}_\kappa(A)$ with the induced relative topology. Moreover, to simplify the notation we denote by $\cong_{\text{RCT}}^\kappa$ and $\sqsubseteq_{\text{RCT}}^\kappa$ the isomorphism and the embeddability relation on RCT_κ , respectively.

THEOREM 16.21. *Let κ be an infinite cardinal and A be an ill-founded subset of \mathbb{R} satisfying the conditions of Definition 16.18.*

- (a) $\cong_{\text{RCT}}^\kappa \leq_{\mathbf{B}}^\kappa \cong^i \upharpoonright \mathfrak{U}_\kappa(A)$ and $\sqsubseteq_{\text{RCT}}^\kappa \leq_{\mathbf{B}}^\kappa \sqsubseteq^i \upharpoonright \mathfrak{U}_\kappa(A)$.
- (b) *The relation $\sqsubseteq^i \upharpoonright \mathfrak{U}_\kappa(A)$ is $\leq_{\mathbf{B}}^\kappa$ -complete for the class of κ -Souslin quasi-orders on Polish or standard Borel spaces.*
- (c) **(AC)** *If $\kappa \leq 2^{\aleph_0}$ and there is an $\mathcal{S}(\kappa)$ -code for κ (which is always the case for $\kappa = \omega$, $\kappa = \omega_1$, and $\kappa = 2^{\aleph_0}$), then $\sqsubseteq^i \upharpoonright \mathfrak{U}_\kappa(A)$ is $\leq_{\mathcal{S}(\kappa)}$ -complete for κ -Souslin quasi-orders on Polish or standard Borel spaces.*
- (d) **(AD + DC)** *If κ is a Souslin cardinal, then $\sqsubseteq^i \upharpoonright \mathfrak{U}_\kappa(A)$ is $\leq_{\mathcal{S}(\kappa)}$ -complete for κ -Souslin quasi-orders on Polish or standard Borel spaces.*
- (e) *Let $W \supseteq \mathbb{R}$ be an inner model and let $\kappa \in \text{Card}^W$. Then $\sqsubseteq^i \upharpoonright \mathfrak{U}_\kappa(A)$ is \leq_W -complete for quasi-orders in $\mathcal{S}_W(\kappa)$.*

PROOF. As observed in Remark 10.1, in all the constructions and results from Sections 10–12 and 14 we could systematically replace combinatorial trees with rooted combinatorial trees. Then it is enough to argue as in the proof of Theorem 16.15 but replacing Remark 16.13 and Lemma 16.14 with Remark 16.19 and Lemma 16.20, respectively. \square

By Theorem 16.21, in all the completeness results from Sections 14, 15, and 16.1 we may systematically replace $\sqsubseteq_{\text{CT}}^\kappa$ with $\sqsubseteq^i \upharpoonright \mathfrak{U}_\kappa$: here is a sample of the statements that one may obtain in this way.

- THEOREM 16.22.** (a) **(AC)** *The relation $\sqsubseteq^i \upharpoonright \mathfrak{U}_{\omega_1}$ is $\leq_{\mathbf{B}}^{\omega_1}$ -complete for Σ_2^1 quasi-orders on Polish or standard Borel spaces. If moreover we assume either $\text{AD}^{\text{L}(\mathbb{R})}$ or $\text{MA} + \neg\text{CH} + \exists a \in {}^\omega\omega$ ($\omega_1^{\text{L}[a]} = \omega_1$), then $\sqsubseteq^i \upharpoonright \mathfrak{U}_{\omega_1}$ is also $\leq_{\Sigma_2^1}$ -complete for Σ_2^1 quasi-orders on Polish or standard Borel spaces.*
- (b) **(AC)** *Assume that $x^\#$ exists for all $x \in {}^\omega\omega$. Then the relation $\sqsubseteq^i \upharpoonright \mathfrak{U}_{\omega_2}$ is $\leq_{\mathbf{B}}^{\omega_2}$ -complete for Σ_3^1 quasi-orders on Polish or standard Borel spaces. In particular, the quotient order of $(\mathcal{Q}, \leq_{\mathbf{B}})$ (definably) embeds into the quotient order of $\sqsubseteq^i \upharpoonright \mathfrak{U}_{\aleph_2}$.*
- (c) **(AC)** *Assume $\text{AD}^{\text{L}(\mathbb{R})}$. Then the relation $\sqsubseteq^i \upharpoonright \mathfrak{U}_{\omega_{r(n)}}$ is $\leq_{\mathbf{B}}^{\omega_{r(n)}}$ -complete for Σ_n^1 quasi-orders on Polish or standard Borel spaces, where $r: \omega \rightarrow \omega$ is as in Theorem 15.8.*
- (d) **(AD + DC)** *For $0 \neq n \in \omega$, let κ_n be such that $\delta_n^1 = \kappa_n^+$ (see Section 15.2.2). Then the relation $\sqsubseteq^i \upharpoonright \mathfrak{U}_{\kappa_n}$ is both $\leq_{\mathbf{B}}^{\kappa_n}$ -complete and $\leq_{\Sigma_n^1}$ -complete for Σ_n^1 quasi-orders on Polish or standard Borel spaces. In particular, the quotient order of $(\mathcal{Q}, \leq_{\mathbf{B}})$ (definably) embeds into the quotient order of $\sqsubseteq^i \upharpoonright \mathfrak{U}_{\aleph_\omega}$.*
- (e) **(AC)** *Assume $\text{AD}^{\text{L}(\mathbb{R})}$. Then the relation $\sqsubseteq^i \upharpoonright \mathfrak{U}_{\delta_1^2}$ is $\leq_{\text{L}(\mathbb{R})}$ -complete for quasi-orders on ${}^\omega 2$ (or on arbitrary Polish or standard Borel spaces in $\text{L}(\mathbb{R})$) belonging to $\Gamma_1^2 := (\Sigma_1^2)^{\text{L}(\mathbb{R})}$.*

- (f) (AC) Assume that there is a Woodin cardinal which is limit of Woodin cardinals. Let M_ω be the minimal model of $\text{AD}_\mathbb{R}$, and let $\kappa := \Theta^{M_\omega} = \Theta_\omega$ (see Section 15.5). Then the relation $\sqsubseteq^i \restriction \mathfrak{U}_\kappa$ is \leq_{M_ω} -complete for quasi-orders on ${}^\omega 2$ (or on arbitrary Polish or standard Borel spaces in M_ω) belonging to M_ω . In fact, for every quasi-order R on ${}^\omega 2$

$$R \in M_\omega \Leftrightarrow R \leq_{M_\omega} \sqsubseteq^i \restriction \mathfrak{U}_\kappa.$$

Moreover, combining Theorem 16.21 with the results from Section 16.1 we get that the quasi-order $\sqsubseteq^i \restriction \mathfrak{U}_\kappa$ is complicated also from the combinatorial point of view.

- THEOREM 16.23. (a) Let $\omega < \kappa \leq 2^{\aleph_0}$ and assume $\text{AC}_\kappa(\mathbb{R})$. Then the relation \sqsubseteq_κ^* on $\mathcal{P}(\kappa)$ of inclusion modulo bounded subsets is $\kappa + 1$ -Borel reducible to $\sqsubseteq^i \restriction \mathfrak{U}_\kappa$. In particular, every partial order P of size κ can be embedded into the quotient order of $\sqsubseteq^i \restriction \mathfrak{U}_\kappa$. Further assuming AC and $2^{\aleph_0} \geq \aleph_n$ (for some $n \in \omega$), we also get that every linear order of size \aleph_{n+1} can be embedded into the quotient order of $\sqsubseteq^i \restriction \mathfrak{U}_{\aleph_n}$.
- (b) Assume $\text{AD} + \text{DC}$ and let κ be a Souslin cardinal. Then every partial order P of size κ can be embedded into the quotient order of $\sqsubseteq^i \restriction \mathfrak{U}_\kappa$.

PROOF. Use again the fact that in Proposition 16.2 and Theorem 16.4 we could have replaced combinatorial trees with rooted combinatorial trees, and then apply Theorem 16.21(a). \square

REMARK 16.24. In Theorems 16.22 and 16.23 we may also consider just isometric embeddability between ultrametric spaces in $\mathfrak{U}_\kappa(A)$ for any ill-founded set of distances A (provided that A is as in Definition 16.18). As already observed, when A is bounded away from 0 then all spaces in $\mathfrak{U}_\kappa(A)$ are discrete (and hence of size κ). This implies that in Theorems 16.22 and 16.23 we may replace the collection of complete ultrametric spaces of density character κ with any subclass of \mathfrak{M}_κ containing one of these $\mathfrak{U}_\kappa(A)$, including the following notable examples (all spaces below are intended to be complete and of density character κ):

- discrete (ultrametric) spaces;
- (ultrametric) spaces of size κ ;
- locally compact (ultrametric) spaces;
- zero-dimensional spaces.

One can also show that \cong^i remains consistently as complicated as possible when restricted to \mathfrak{U}_κ , and even to $\mathfrak{U}_\kappa \cap \mathfrak{D}_\kappa$.

COROLLARY 16.25. Assume $V = L$, and let $\kappa = \lambda^+$ be such that $\lambda^\omega = \lambda$. Then the relation $\cong^i \restriction \mathfrak{U}_\kappa(A)$ is $\leq_{\mathbf{B}}^\kappa$ -complete for the collection of all κ -analytic equivalence relations on standard Borel κ -spaces whenever A is a subset of \mathbb{R} satisfying the conditions of Definition 16.18 and containing a strictly decreasing chain of length $2 \cdot \omega + 1$ (with respect to the usual ordering of \mathbb{R}).

A complete and detailed proof of Corollary 16.25 is given in [MR]: here we just describe the relevant main construction for the sake of completeness, referring the reader to [MR, Sections 3 and 4] for more details.

SKETCH OF THE PROOF. Call **(set-theoretic) tree** any partially ordered set $T = (T, \preceq)$ with a minimum and such that for every $t \in T$ the set $\text{pred}_T(t) := \{s \in T \mid s \preceq t\}$ of predecessors of t is well-ordered by \preceq . By [HK15, Theorem 2.10], under our assumptions the relation of isomorphism between trees of size κ and height $2 \cdot \omega + 1$ (i.e. such that $\text{pred}_T(t)$ has length at most $2 \cdot \omega + 1$ for every $t \in T$) is $\leq_{\mathbf{B}}^\kappa$ -complete for κ -analytic equivalence relations on ${}^\kappa 2$, and hence for κ -analytic equivalence relations on arbitrary standard Borel κ -spaces. It is thus enough to provide a reduction from such isomorphism relation to $\cong^i \restriction \mathfrak{U}_\kappa(A)$, and this can be obtained by adapting the definition of the map $\theta_U^{\vec{r}}$ from (16.5) to this new context. Fix a strictly decreasing sequence $\vec{r} = \langle r_\alpha \mid \alpha \leq 2 \cdot \omega \rangle$ in A . Given a tree $T = (\kappa, \preceq)$ as above, define a metric $d_T^{\vec{r}}$ on κ

by setting $d_T^{\vec{r}}(\alpha, \beta) := r_{\gamma(\alpha, \beta)}$, where γ is the order type with respect to \preceq of the well-ordered subset $\text{pred}_T(\alpha) \cap \text{pred}_T(\beta)$ of T . It is straightforward to check that $d_T^{\vec{r}}$ is indeed an ultrametric, and that the map sending T to the natural code $u_T^{\vec{r}} \in \mathfrak{U}_\kappa(A)$ for the completion $U_T^{\vec{r}} = (U_T, d_T^{\vec{r}})$ of the space $(\kappa, d_T^{\vec{r}})$ is a τ_b -continuous function from the subspace of $\text{Mod}_{\mathcal{L}}^\kappa$ consisting of all trees as above to $\mathfrak{U}_\kappa(A) \subseteq \mathfrak{M}_\kappa$. Arguing as in Lemma 16.20, one then check that the map $T \mapsto u_T^{\vec{r}}$ reduces \cong to \cong^i , hence we are done. \square

Since in Corollary 16.25 the set A can be chosen to be bounded away from 0, Remark 16.24 applies here as well: as a consequence, the isometry relation on any of the classes of metric spaces mentioned in that remark is consistently as complicated as possible.

Whether $\sqsubseteq^i \upharpoonright \mathfrak{U}_\kappa$ may consistently have maximal complexity seems to be open: the main problem is that we do not know e.g. whether $\sqsubseteq_{\text{RCT}}^\kappa$ can be $\leq_{\mathbf{B}}^\kappa$ -complete for κ -analytic quasi-orders on ${}^\kappa 2$ (equivalently, whether it has the same complexity as $\sqsubseteq_{\text{GRAPH}}^\kappa$). However, we can at least observe that $\cong_{\text{GRAPH}}^\kappa$ and $\sqsubseteq_{\text{GRAPH}}^\kappa$ are always upper bounds for the complexity of $\cong^i \upharpoonright \mathfrak{U}_\kappa$ and $\sqsubseteq^i \upharpoonright \mathfrak{U}_\kappa$, respectively.

THEOREM 16.26. *Let κ be any infinite cardinal. Then $\cong^i \upharpoonright \mathfrak{U}_\kappa \leq_{\mathbf{B}}^\kappa \cong_{\text{GRAPH}}^\kappa$ and $\sqsubseteq^i \upharpoonright \mathfrak{U}_\kappa \leq_{\mathbf{B}}^\kappa \sqsubseteq_{\text{GRAPH}}^\kappa$.*

PROOF. Argue as in the proof of [GK03, Theorem 4.4] and use the fact that $\cong \upharpoonright \text{Mod}_{\mathcal{L}}^\kappa \leq_{\mathbf{B}}^\kappa \cong_{\text{GRAPH}}^\kappa$ and $\sqsubseteq \upharpoonright \text{Mod}_{\mathcal{L}}^\kappa \leq_{\mathbf{B}}^\kappa \sqsubseteq_{\text{GRAPH}}^\kappa$ for any countable language \mathcal{L} by [Hod93, Theorem 5.5.1] (see also the proof of Corollary 16.16(a)). \square

Since Theorems 16.17 and 16.26 are proved in ZF, it follows that their conclusions are true in $L(\mathbb{R})$. This shows that in models of $\text{AD}^{L(\mathbb{R})}$ the relation of isometric bi-embeddability between ultrametric or discrete complete metric spaces of uncountable density character is way more complicated than the isometry relation on the same class: in fact, by Theorem 16.22(e) every equivalence relation in $\mathbf{\Gamma}_1^2$ (a quite large boldface pointclass in V which includes e.g. all projective levels) is $L(\mathbb{R})$ -reducible to the isometric bi-embeddability relation on $\mathfrak{U}_\kappa \cap \mathfrak{D}_\kappa$ (for a suitable cardinal κ), while by Example 1.10 (see [Hjo00, Theorem 9.18]) and Theorems 16.17 and 16.26 there are Σ_1^1 equivalence relations E such that $E \not\leq_{L(\mathbb{R})} \cong^i \upharpoonright \mathfrak{D}_\kappa$ and $E \not\leq_{L(\mathbb{R})} \cong^i \upharpoonright \mathfrak{U}_\kappa$ for every $\kappa \in (\text{Card})^{L(\mathbb{R})}$, and therefore for every $\kappa \in \text{Card}$.

Combining this observation with Corollary 16.25 we further get the following interesting independence result.

COROLLARY 16.27. *It is independent of $\text{ZF} + \text{DC}$ whether for $\kappa = \omega_2$ (or for any κ which is a successor of some λ satisfying $\lambda^\omega = \lambda$) the relation of isometric bi-embeddability between ultrametric (respectively, discrete) complete metric spaces of density character κ is $\leq_{\mathbf{B}}^\kappa$ -reducible to the isometry relation on the same class of spaces.*

PROOF. Since the relation of isometric bi-embeddability is a κ -analytic equivalence relation on the standard Borel κ -space \mathfrak{M}_κ , under $V = L$ it is $\leq_{\mathbf{B}}^\kappa$ -reducible to $\cong^i \upharpoonright (\mathfrak{U}_\kappa \cap \mathfrak{D}_\kappa)$ by Corollary 16.25 (and the ensuing remark). On the other hand, by the observation preceding this corollary we get that under $\text{AD} + V = L(\mathbb{R})$ there are Σ_1^1 equivalence relations on \mathbb{R} which are not reducible (even without definability conditions on the reductions that may be used) to $\cong^i \upharpoonright \mathfrak{D}_\kappa$ or to $\cong^i \upharpoonright \mathfrak{U}_\kappa$, while all such relations are $\leq_{\mathbf{B}}^\kappa$ -reducible to the isometric bi-embeddability relation on $\mathfrak{U}_\kappa \cap \mathfrak{D}_\kappa$ by Theorem 16.21(b) (and Remark 16.24); thus in this case the isometric bi-embeddability relation is not $\leq_{\mathbf{B}}^\kappa$ -reducible to $\cong^i \upharpoonright \mathfrak{D}_\kappa$ or $\cong^i \upharpoonright \mathfrak{U}_\kappa$. \square

We conclude this section by observing that in Theorem 16.21 we could also systematically replace all occurrences of $\mathfrak{U}_\kappa(A)$ with the smaller $\mathfrak{U}_\kappa^*(A)$. This is because one can find (for any

ill-founded A satisfying the conditions of Definition 16.18) a modification $\theta_{U,A}^*: \text{RCT}_\kappa \rightarrow \mathfrak{U}_\kappa^*(A)$ of the map $\theta_U^{\vec{r}}$ from (16.5) such that:

- (1) the map $\theta_{U,A}^*$ is continuous when both RCT_κ and $\mathfrak{U}_\kappa^*(A)$ are endowed with the same topology τ_p or τ_b ;
- (2) the map $\theta_{U,A}^*$ simultaneously reduces isomorphism to isometry and embeddability to isometric embeddability.

In fact, given such an $A \subseteq \mathbb{R}$ fix a strictly decreasing sequence $\vec{r} = \langle r_n \mid n \in \omega \rangle$ of points from A such that there is $\bar{r} \in A$ with $\bar{r} > r_0$. Given $G \in \text{RCT}_\kappa$, consider the space $U_G^{\vec{r}}$ coded by $\theta_U^{\vec{r}}(G) = u_G^{\vec{r}} \in \mathfrak{U}_\kappa(A)$, and notice that since G has size κ (in particular, it has vertices distinct from its root) each point of $U_G^{\vec{r}}$ realizes the distance r_0 . Then let U_G^A be the disjoint union of $U_G^{\vec{r}}$ and $U(A \setminus \{r_0\})$ (where $U(A \setminus \{r_0\}) = (U_{A \setminus \{r_0\}}, d_{A \setminus \{r_0\}})$ is defined as in (16.4)), and endow it with the metric d_G^A extending both $d_G^{\vec{r}}$ and $d_{A \setminus \{r_0\}}$ obtained by setting for every $x \in U_G^{\vec{r}}$, $r \in A \setminus \{r_0\}$, and $\alpha < \kappa$

$$d_G^A(x, (r, \alpha)) := \max\{\bar{r}, r\}.$$

A code $u_G^A \in \mathfrak{U}^*(A)$ for U_G^A can be obtained in a continuous-in- G way by e.g. copying the code $u_G^{\vec{r}}$ on the odd ordinal numbers and (a code for) the space $U(A \setminus \{r_0\})$ on the even ordinal numbers — we leave to the reader to carry out the details of such coding procedure. Consider the map $\theta_{U,A}^*: \text{RCT}_\kappa \rightarrow \mathfrak{U}_\kappa^*(A) \subseteq \mathfrak{M}_\kappa$ defined by setting $\theta_{U,A}^*(G) := u_G^A$ for every $G \in \text{RCT}_\kappa$. By the coding procedure briefly sketched above, condition (1) is satisfied. To show that also (2) is satisfied, first notice that by Lemma 16.20 we only need to check that for all $G, G' \in \text{RCT}_\kappa$

$$\theta_U^{\vec{r}}(G) \cong^i \theta_U^{\vec{r}}(G') \Leftrightarrow \theta_{U,A}^*(G) \cong^i \theta_{U,A}^*(G') \quad \text{and} \quad \theta_U^{\vec{r}}(G) \sqsubseteq^i \theta_U^{\vec{r}}(G') \Leftrightarrow \theta_{U,A}^*(G) \sqsubseteq^i \theta_{U,A}^*(G').$$

The forward direction is obvious, so assume that $\theta_{U,A}^*(G) \cong^i \theta_{U,A}^*(G')$ (respectively, $\theta_{U,A}^*(G) \sqsubseteq^i \theta_{U,A}^*(G')$) and let φ be an isometry (respectively, an isometric embedding) between the spaces U_G^A and $U_{G'}^A$ coded by $\theta_{U,A}^*(G)$ and $\theta_{U,A}^*(G')$. Then since by construction the points in $U_G^{\vec{r}} \subseteq U_G^A$ are the unique realizing the distance r_0 in U_G^A (and the same is true when replacing G with G'), then $\varphi \upharpoonright U_G^{\vec{r}}$ is an isometry (respectively, an isometric embedding) between $U_G^{\vec{r}}$ and $U_{G'}^{\vec{r}}$, so that $\theta_U^{\vec{r}}(G) \cong^i \theta_U^{\vec{r}}(G')$ (respectively, $\theta_U^{\vec{r}}(G) \sqsubseteq^i \theta_U^{\vec{r}}(G')$).

A similar construction shows that we may as well replace $\mathfrak{U}_\kappa(A)$ with $\mathfrak{U}_\kappa^*(A)$ in Corollary 16.25.

16.4. Linear isometry and linear isometric embeddability between Banach spaces of density κ . Let $\mathbf{c}_0 = (c_0, \|\cdot\|_\infty)$ be the separable (real) Banach space of vanishing ω -sequences of reals endowed with the usual pointwise operations and the sup norm $\|\cdot\|_\infty$. Building on previous work by Louveau and Rosendal [LR05], in [CMMR13, Section 5.6] it was defined a Borel map simultaneously reducing isomorphism and embeddability between countable graphs to, respectively, linear isometry and linear isometric embeddability between separable Banach spaces isomorphic to \mathbf{c}_0 . We are now going to adapt such construction to the uncountable context in order to study the complexity of the relations of linear isometry \cong^{li} and linear isometric embeddability \sqsubseteq^{li} between non-separable Banach spaces. To this aim we first have to define the right analogue of the space \mathbf{c}_0 .

DEFINITION 16.28. Given an infinite cardinal κ , let $c_0^\kappa \subseteq {}^\kappa\mathbb{R}$ be the collection of all κ -sequences $x = \langle x_\alpha \mid \alpha < \kappa \rangle$ of reals such that for all $\varepsilon \in \mathbb{R}^+$ we have $|x_\alpha| < \varepsilon$ for all but finitely many $\alpha < \kappa$.

The Banach space $\mathbf{c}_0^\kappa = (c_0^\kappa, \|\cdot\|_\infty)$ is then obtained by endowing c_0^κ with the usual pointwise operations and the sup norm $\|\cdot\|_\infty$.

In particular, $\mathbf{c}_0^\omega = \mathbf{c}_0$. A basis $\langle e_\alpha \mid \alpha < \kappa \rangle$ for \mathbf{c}_0^κ is obtained by letting e_α be the κ -sequence with value 1 on coordinate α and 0 elsewhere; then each element $x \in \mathbf{c}_0^\kappa$ may be uniquely written as $\sum_\alpha x_\alpha e_\alpha$. Notice that \mathbf{c}_0^κ has density κ (a dense subset is given by the collection of all $x \in \mathbf{c}_0^\kappa \cap {}^\kappa\mathbb{Q}$ having only finitely many non-null coordinates), and that each $x \in \mathbf{c}_0^\kappa$ has countable support, that is it has at most countably many non-null coordinates.

Given any graph G on κ , let $\mathbf{X}_G = (c_0^\kappa, \|\cdot\|_G)$ be the (real) Banach space on c_0^κ equipped with the pointwise operations and the norm defined by

$$\left\| \sum_\alpha x_\alpha e_\alpha \right\|_G := \sup \left\{ |x_i| + \frac{|x_j|}{3 - \chi_G(i,j)} \mid i \neq j \in \kappa \right\},$$

where $\chi_G: \kappa \times \kappa \rightarrow \{0,1\}$ is the characteristic function of the graph relation of G . It is easy to verify that $\|\cdot\|_G$ is equivalent to $\|\cdot\|_\infty$, as $\|\sum_\alpha x_\alpha e_\alpha\|_\infty \leq \|\sum_\alpha x_\alpha e_\alpha\|_G \leq \frac{3}{2} \|\sum_\alpha x_\alpha e_\alpha\|_\infty$. Moreover, \mathbf{X}_G has density κ and thus it can be coded as in (7.7) by an element $x_{\mathbf{X}_G} = (x_{\mathbf{X}_G}^+, x_{\mathbf{X}_G}^\mathbb{Q}, x_{\mathbf{X}_G}^{\|\cdot\|})$ of the standard Borel κ -space \mathfrak{B}_κ introduced in Section 7.2.4.

REMARK 16.29. The map

$$(16.6) \quad \theta_B: \text{Mod}_{\text{GRAPH}}^\kappa \rightarrow \mathfrak{B}_\kappa, \quad G \mapsto x_{\mathbf{X}_G}$$

is continuous when both $\text{Mod}_{\text{GRAPH}}^\kappa$ and \mathfrak{B}_κ are endowed with the same topology τ_p or τ_b .

The proof of the next lemma is identical⁴⁸ to the one of [CMMR13, Lemma 5.20], so we omit it here.

LEMMA 16.30. *The map θ_B from (16.6) simultaneously reduces \cong to \cong^{li} and \sqsubseteq to \sqsubseteq^{li} .*

Arguing as in the case of metric spaces one can straightforwardly check that Remark 16.29 and Lemma 16.30 yield the following lower bounds for the complexity of $\cong^{li} \upharpoonright \mathfrak{B}_\kappa$ and $\sqsubseteq^{li} \upharpoonright \mathfrak{B}_\kappa$. (As usual, in what follows both $\text{Mod}_{\text{GRAPH}}^\kappa$ and \mathfrak{B}_κ are endowed with the bounded topology.)

THEOREM 16.31. *Let κ be an infinite cardinal.*

- (a) $\cong_{\text{GRAPH}}^\kappa \leq_{\mathbf{B}}^\kappa \cong^{li} \upharpoonright \mathfrak{B}_\kappa$ and $\sqsubseteq_{\text{GRAPH}}^\kappa \leq_{\mathbf{B}}^\kappa \sqsubseteq^{li} \upharpoonright \mathfrak{B}_\kappa$.
- (b) *The relation $\sqsubseteq^{li} \upharpoonright \mathfrak{B}_\kappa$ is $\leq_{\mathbf{B}}^\kappa$ -complete for the class of κ -Souslin quasi-orders on Polish or standard Borel spaces.*
- (c) (AC) *If $\kappa \leq 2^{\aleph_0}$ and there is an $\mathbf{S}(\kappa)$ -code for κ (which is always the case for $\kappa = \omega$, $\kappa = \omega_1$, and $\kappa = 2^{\aleph_0}$), then $\sqsubseteq^{li} \upharpoonright \mathfrak{B}_\kappa$ is $\leq_{\mathbf{S}(\kappa)}$ -complete for κ -Souslin quasi-orders on Polish or standard Borel spaces.*
- (d) (AD + DC) *If κ is a Souslin cardinal, then $\sqsubseteq^{li} \upharpoonright \mathfrak{B}_\kappa$ is $\leq_{\mathbf{S}(\kappa)}$ -complete for κ -Souslin quasi-orders on Polish or standard Borel spaces.*
- (e) *Let $W \supseteq \mathbb{R}$ be an inner model and let $\kappa \in \text{Card}^W$. Then $\sqsubseteq^{li} \upharpoonright \mathfrak{B}_\kappa$ is \leq_W -complete for quasi-orders in $\mathbf{S}_W(\kappa)$.*

Theorem 16.31 shows in particular that in all the completeness results from Sections 14, 15 and 16.1 we may further replace $\sqsubseteq_{\text{CT}}^\kappa$ with $\sqsubseteq^{li} \upharpoonright \mathfrak{B}_\kappa$. To see a sample of the statements that one may obtain in this way just systematically replace $\sqsubseteq^i \upharpoonright \mathfrak{U}_\kappa$ with $\sqsubseteq^{li} \upharpoonright \mathfrak{B}_\kappa$ in Theorems 16.22 and 16.23. Finally, from Theorem 16.31(a) we obtain that, as for the case of \cong^i and \sqsubseteq^i , also the relations \cong^{li} and \sqsubseteq^{li} are consistently as complicated as possible. (The proof is identical to the one of Corollary 16.16, so we omit it here.)

⁴⁸It is enough to check that all the properties of the norm $\|\cdot\|_G$ used in the original proof are maintained when passing to the uncountable context, including e.g. the fact that the sup in the definition of $\|\cdot\|_G$ is attained, or the fact that when $\|\sum_\alpha x_\alpha e_\alpha\|_G \geq \varepsilon$ for some $\varepsilon \in \mathbb{R}^+$, then there are only finitely many coordinates $i, j \in \kappa$ for which $|x_i| + \frac{|x_j|}{3 - \chi_G(i,j)} > \varepsilon$.

- COROLLARY 16.32. (a) Assume $V = L$, and let $\kappa = \lambda^+$ be such that $\lambda^\omega = \lambda$. Then the relation $\cong^{li} \restriction \mathfrak{B}_\kappa$ is $\leq_{\mathbf{B}}^\kappa$ -complete for the collection of all κ -analytic equivalence relations on standard Borel κ -spaces.
- (b) Assume AC and let κ be a weakly compact⁴⁹ cardinal. Then the relation $\sqsubseteq^{li} \restriction \mathfrak{B}_\kappa$ is $\leq_{\mathbf{B}}^\kappa$ -complete for the collection of all κ -analytic quasi-orders on standard Borel κ -spaces.

⁴⁹[MMR16] shows that the same is true for any cardinal satisfying the equality $\kappa^{<\kappa} = \kappa$.

Indexes

Here you will find two indexes, one for the **concepts** and one for the **symbols**. The index of symbols is essentially divided in two parts: in the first part we listed in the order they appear in the text all those symbols (such as \simeq) that cannot easily be placed in alphabetical order, and in the second part we list lexicographically all the other symbols (such as Σ_T).

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